4. Mobius Transformations and Conformal Models of $\mathbb{H}^n$.

Def An injective linear function $\varphi: V \to W$ between inner product spaces is angle preserving if

$$\frac{\varphi(x)}{\|\varphi(x)\|} \cdot \frac{\varphi(y)}{\|\varphi(y)\|} = \frac{x}{\|x\|} \cdot \frac{y}{\|y\|}$$

for all non-zero $x, y \in V$.

Lemma 4.1 Let $\varphi: V \to W$ be an injective linear function between finite dimensional inner product spaces. Then φ is angle preserving if and only if there is an $r > 0$ such that

$$\varphi(x) \cdot \varphi(y) = r^2 (x \cdot y)$$

for all $x, y \in V$.

Proof First assume φ is angle preserving. Let $u_1, \ldots, u_n$ be an orthonormal basis for V. Let $r = \|\varphi(u_1)\|$.

Observe that if $x, y \in V$ and $x \cdot y = 0$, then $\varphi(x) \cdot \varphi(y) = 0$.

For $2 \le j \le n$, $(u_j - u_1)\cdot(u_j + u_1) = \|u_j\|^2 - \|u_1\|^2$ $= 1-1 = 0$. Hence

$$\|\varphi(u_j)\|^2 - r^2 = \|\varphi(u_j)\|^2 - \|\varphi(u_1)\|^2 =$$
$$(\varphi(u_j) - \varphi(u_1))\cdot(\varphi(u_j) + \varphi(u_1)) =$$
$$(\varphi(u_j - u_1))\cdot(\varphi(u_j + u_1)) = 0.$$

Thus, $\|\varphi(u_j)\| = r$ for $1 \le j \le n$.

Let $x, y \in V$. Write $x = \sum_{i=1}^{n} a_i u_i$ and $y = \sum_{j=1}^{n} b_j u_j$. Since $\{u_1, \dots, u_n\}$ are orthogonal, then so are $\{\varphi(u_1), \dots, \varphi(u_n)\}$. Hence,

$$\varphi(x)\cdot\varphi(y) = \left(\sum_{i=1}^{n} a_i \varphi(u_i)\right)\cdot\left(\sum_{j=1}^{n} b_j \varphi(b_j)\right) =$$
$$\sum_{i=1}^{n} a_i b_i \|\varphi(u_i)\|^2 = r^2 \sum_{i=1}^{n} a_i b_i =$$
$$r^2\left(\sum_{i=1}^{n} a_i u_i\right)\cdot\left(\sum_{j=1}^{n} b_j u_j\right) = r^2(x\cdot y).$$

Now assume there is an $r > 0$ such that $\varphi(x)\cdot\varphi(y) = r^2(x\cdot y)$ for all $x, y \in V$. Then for each $x \in V$,

$\|\varphi(x)\| = r\|x\|$. Thus for all non-zero $x, y \in V$

$$\frac{\varphi(x)\cdot\varphi(y)}{\|\varphi(x)\|\,\|\varphi(y)\|} = \frac{r^2(x\cdot y)}{(r\|x\|)(r\|y\|)} = \frac{x\cdot y}{\|x\|\,\|y\|}.$$

Thus φ is angle preserving. $\square$

<u>Theorem 4.2</u> Let $\varphi : V \to W$ be an injective linear function between finite dimensional vector spaces. Then φ is angle preserving if and only if there is an orthogonal function $\psi : V \to W$ ~~such tha~~ (i.e. $\psi(x)\cdot\psi(y) = x\cdot y$) such that $\varphi(x) = r\psi(x)$ for some $r > 0$.

<u>Proof</u> First assume φ is angle preserving. Lemma 4.1 provides an $r > 0$ such that $\varphi(x)\cdot\varphi(y) = r^2(x\cdot y)$ for all $x, y \in V$. Define $\psi : V \to W$ by $\psi(x) = \frac{1}{r}\varphi(x)$. ψ is clearly an injective linear function. For $x, y \in V$,

$$\psi(x)\cdot\psi(y) = \left(\tfrac{1}{r}\varphi(x)\right)\cdot\left(\tfrac{1}{r}\varphi(y)\right) = \frac{\varphi(x)\cdot\varphi(y)}{r^2} = x\cdot y$$

Thus ψ is orthogonal and $\varphi = r\psi$.

Now assume there is an orthogonal function $\psi : V \to W$ such that $\varphi = r\psi$ for some $r > 0$. Then for $x, y \in V$,

$$\varphi(x)\cdot\varphi(y) = (r\psi(x))\cdot(r\psi(y)) = r^2(\psi(x)\cdot\psi(y)) = r^2(x\cdot y).$$

Hence, Lemma 4.1 implies φ is angle preserving. $\square$

<u>Def</u> A continuously differentiable embedding $f : M \to N$ between Riemannian manifolds is <u>conformal</u> if $df_x : T_x(M) \to T_{f(x)}(N)$ is angle preserving for each $x \in M$.

<u>Observation</u> Since orthogonal maps between inner product spaces are angle preserving, then isometric embeddings between Riemannian manifolds (each df_x is orthogonal) are conformal.

Recall that for a unit vector $u \in \mathbb{E}^n$ and $a \in \mathbb{R}$, the <u>hyperplane</u> $P(u,a)$ is defined to be

$$P(u,a) = \{x \in \mathbb{E}^n : x \cdot u = a\}$$

and the <u>reflection</u> $Z_{u,a}$ in $P(u,a)$ is define by

$$Z_{u,a}(x) = x - 2(x \cdot u - a)u.$$

Recall that $Z_{u,a}$ is an isometry of $\mathbb{E}^n$, $Z_{ua}^{-1} = Z_{ua}$, and the fixed point set of Z_{ua} is P_{ua}.

We will now define inversion in a sphere. It is an analogue of a reflection, but its fixed point set is a sphere rather than a hyperplane.

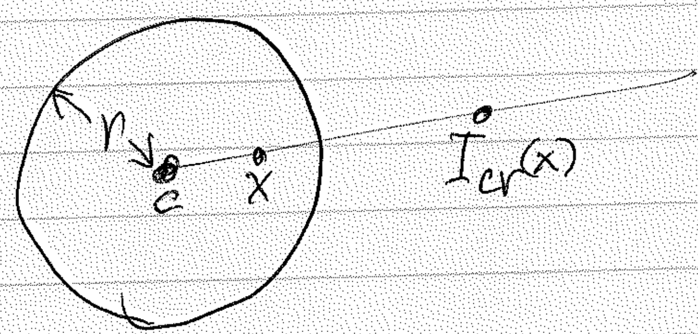
<u>Def</u> Let $c \in \mathbb{E}^n$ and $r > 0$. Define the <u>hypersphere of radius r centered at c</u> to be

$$S(c,r) = \{x \in \mathbb{E}^n : \|x - c\| = r\}.$$

Define the <u>inversion</u> $I_{c,r} : \mathbb{E}^n - \{c\} \to \mathbb{E}^n - \{c\}$ <u>in $S(c,r)$</u> by the equation

$$I_{c,r}(x) = \frac{r^2}{\|x-c\|^2}(x-c) + c.$$

Observe that $\|I_{cr}(x) - c\| \, \|x-c\| = r^2$ for each $x \in \mathbb{E}^n - \{c\}$.



Theorem 4.3. Let $c \in \mathbb{E}^n$ and $r > 0$.

a) $I_{c,r} : (\mathbb{E}^n - \{c\}) \to (\mathbb{E}^n - \{c\})$ is a conformal diffeomorphism. In fact, for $x \in \mathbb{E}^n - \{c\}$,

$$d(I_{cr})_x = \frac{r^2}{\|x-c\|^2} Z_{u,0}$$

where $u = \frac{x-c}{\|x-c\|}$.

b) $I_{cr}^{-1} = I_{cr}$.

c) $I_{cr}(x) = x$ if and only if $x \in S(c,r)$.

Remark We observe that $d(I_{cr})_x = \frac{r^2}{\|x-c\|^2} Z_{u,0}$ is a reasonable guess. First, if I_{cr} is conformal, then $d(I_{cr})_x$ is a scalar multiple of an isometry. Since $d(I_{cr})_x$ is linear, the isometry fixes 0. Near x, I_{cr} flips motion in the direction of $u = (x-c)/\|x-c\|$ to motion in the direction of $-u$. So $Z_{u,0}$ is a reasonable candidate for the isometry. Since $\|I_{cr}(x)-c\|\,\|x-c\| = r^2$, then $\|I_{cr}(x) - c\| = \left(\frac{r^2}{\|x-c\|^2}\right)\|x-c\|$. Thus $\frac{r^2}{\|x-c\|^2}$ is a reasonable candidate for the scale factor.

<u>Proof</u> The proofs of b) and c) are simple calculation that we omit. We prove a). Let $x \in \mathbb{E}^n - \{c\}$ and $v \in \mathbb{E}^n$. ~~Then~~ Let $u = (x-c)/\|x-c\|$. Then

$$d(I_{cr})_x(v) = \lim_{t\to 0} \frac{I_{cr}(x+tv) - I_{cr}(x)}{t} =$$

$$\lim_{t\to 0} \frac{1}{t}\left[\frac{r^2}{\|x+tv-c\|^2}(x+tv-c) - \frac{r^2}{\|x-c\|^2}(x-c)\right] =$$

$$\lim_{t\to 0} \frac{r^2}{t}\left[\frac{\|x-c\|^2(x+tv-c) - \|x+tv-c\|^2(x-c)}{\|x+tv-c\|^2\,\|x-c\|^2}\right] =$$

$$\lim_{t\to 0} \frac{r^2}{t}\left[\frac{\cancel{\|x-c\|^2(x-c)} + \|x-c\|^2 tv - \cancel{\|x-c\|^2(x-c)} - 2((x-c)\cdot(tv))(x-c) - t^2\|v\|^2(x-c)}{\|x+tv-c\|^2\,\|x-c\|^2}\right.$$

$$= \lim_{t\to 0} r^2 \left[\frac{\|x-c\|^2 v - 2((x-c)\cdot v)(x-c) - t\|v\|^2(x-c)}{\|x+tv-c\|^2 \|x-c\|^2} \right] =$$

$$r^2 \frac{\|x-c\|^2 v - 2((x-c)\cdot v)(x-c)}{\|x-c\|^4} =$$

$$\frac{r^2}{\|x-c\|^2}\left(v - 2\left(\left(\frac{x-c}{\|x-c\|}\right)\cdot v\right)\left(\frac{x-c}{\|x-c\|}\right)\right) =$$

$$\frac{r^2}{\|x-c\|^2}(v - 2(v\cdot u)u) = \frac{r^2}{\|x-c\|^2} Z_{u,0}(v).$$

Thus, $d(I_{cr})_x = \frac{r^2}{\|x-c\|^2} Z_{u,0}$.

Thus, I_{cr} is continuously differentiable. Since $I_{cr}^{-1} = I_{cr}$, then I_{cr} is a diffeomorphism from $E^n - \{c\}$ to itself. Since $d(I_{cr})x$ is a scalar multiple of an orthogonal map, it is angle preserving by Theorem 4.2. Thus, I_{cr} is conformal. $\square$

We give a slightly incorrect statement of a theorem of Liouville (without proof):

For $n \geq 3$, if $f: U \to V$ is a conformal diffeomorphism between open subset of $\mathbb{E}^n$, then f is a finite composition of reflections in hyperplanes and inversions in hyperspheres.

The flaw in this statement is illustrated by the following example.

Example Define the dilation $D_{c,a}: \mathbb{E}^n \to \mathbb{E}^n$ where $c \in \mathbb{E}^n$ and $a > 0$ by

$$D_{c,a}(x) = a(x-c) + c.$$

Then $d(D_{c,a})_x = a(\mathrm{id}_{\mathbb{E}^n})$, a scalar multiple of an orthogonal map. Hence, each $d(D_{c,a})_x$ is angle preserving. So D_{ca} is conformal. D_{ca} can easily be expressed as a composition of two inversions:

$$D_{ca} = I_{cs} \circ I_{cr}$$

whenever $s/r = \sqrt{a}$. However, this equation fails at $x = c$ because $I_{cs} \circ I_{cr}$ isn't defined at $x = c$. The fact that inversions aren't defined on all of $\mathbb{E}^n$ presents an obstacle to representing conformal diffeomorphisms as compositions of reflections and inversions. This obstacle can be overcome by enlarging $\mathbb{E}^n$ to its one-point compactification $\hat{\mathbb{E}}^n = \mathbb{E}^n \cup \{\infty\}$ and extending each inversion I_{cr} to a homeomorphism of $\hat{\mathbb{E}}^n$ which interchanges c and ∞.

<u>Def</u> Let ∞ be a point not belonging to $\mathbb{E}^n$. Let $\hat{\mathbb{E}}^n = \mathbb{E}^n \cup \{\infty\}$. Endow $\hat{\mathbb{E}}^n$ with the topology whose elements are either (standard) open subsets of $\mathbb{E}^n$ or sets of the form $\hat{\mathbb{E}}^n - C$ where C is a compact subset of $\mathbb{E}^n$. $\hat{\mathbb{E}}^n$ is called the <u>one-point compactification</u> of $\mathbb{E}^n$. It is straight forward to prove that $\hat{\mathbb{E}}^n$ is homeomorphic to S^n and that the inclusion of $\mathbb{E}^n$ in $\hat{\mathbb{E}}^n$ is a homeomorphism of $\mathbb{E}^n$ onto $\hat{\mathbb{E}}^n - \{\infty\}$.

<u>Def</u> Let $u \in \mathbb{E}^n$ be a unit vector and let $a \in \mathbb{R}$. The set $P(u,a) = \{x \in \mathbb{E}^n : x \cdot u = a\}$ is called a <u>hyperplane</u> in $\mathbb{E}^n$. Define

$$\hat{P}(u,a) = P(u,a) \cup \{\infty\} \subset \hat{\mathbb{E}}^n$$

and call $\hat{P}(u,a)$ an <u>extended hyperplane</u> in $\hat{E}^n$. $\hat{P}(u,a)$ is homeomorphic to S^{n-1}. The reflection $Z_{ua}: E^n \to E^n$ is defined by

$$Z_{ua}(x) = x - 2(x \cdot u - a)u$$

Define the <u>extended reflection</u> $\hat{Z}_{ua}: \hat{E}^n \to \hat{E}^n$ by $\hat{Z}_{u,a} | E^n = Z_{ua}$ and $\hat{Z}_{ua}(\infty) = \infty$.

<u>Def</u> Let $c \in E^n$ and $r > 0$. The set $S(c,r) = \{x \in E^n : \|x - c\| = r\}$ is called a <u>hypersphere</u> in E^n. The inversion $I_{cr}: E^n - \{c\} \to E^n - \{c\}$ is defined by

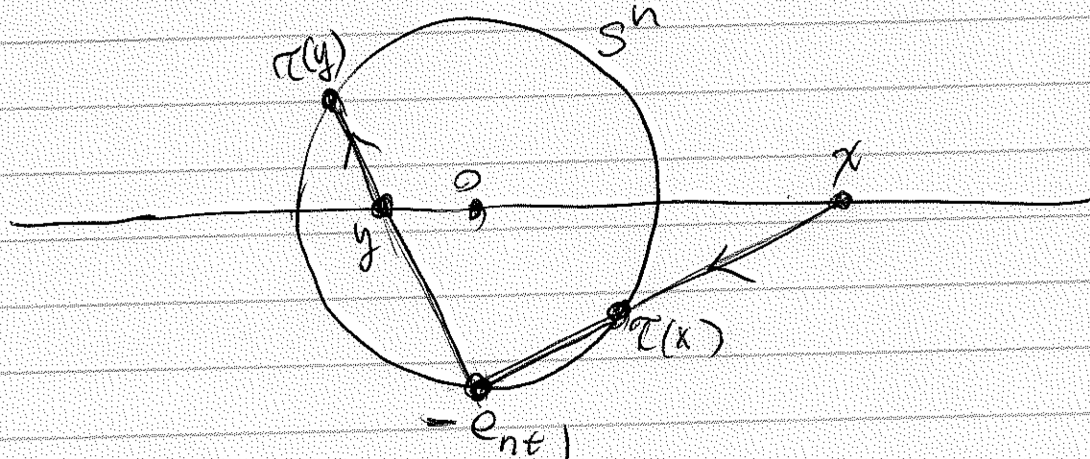
$$I_{cr}(x) = \frac{r^2}{\|x-c\|^2}(x-c) + c$$

Define the <u>extended inversion</u> $\hat{I}_{cr}: \hat{E}^n \to \hat{E}^n$ by $\hat{I}_{cr} | E^n - \{c\} = I_{cr}$, $\hat{I}_{cr}(c) = \infty$ and $\hat{I}_{cr}(\infty) = c$.

<u>Lemma 4.4.</u> a) $\hat{E}^n$ is homeomorphic to S^n, ~~and~~ each $\hat{P}(u,a)$ is homeomorphic to S^{n-1}.
b) Each extended reflection and each extended inversion is a homeomorphism of $\hat{E}^n$
c) $\hat{Z}_{ua}(x) = x$ if and only if $x \in \hat{P}(u,a)$, and $\hat{I}_{cr}(x) = x$ if and only if $x \in S(c,r)$.
d) $\hat{Z}_{ua}^{-1} = \hat{Z}_{ua}$ and $\hat{I}_{cr}^{-1} = \hat{I}_{cr}$.

<u>Proof of a)</u> For $x=(x_1,\dots,x_n)\in\mathbb{E}^n$, let $\bar{x}=(x_1,\dots,x_n,0)\in\mathbb{E}^{n+1}$. The <u>inverse of stereographic projection</u> is a homeomorphism $\tau:\mathbb{E}^n\to S^n-\{-e_{n+1}\}$ defined by

$$\tau(x)=\frac{2}{1+\|x\|^2}(\bar{x}+e_{n+1})-e_{n+1}$$



($\tau^{-1}:S^n-\{-e_{n+1}\}\to\mathbb{E}^n$ satisfies

$$\tau^{-1}(y)=\frac{y-y_{n+1}e_{n+1}}{y_{n+1}+1}.\text{)}$$

Extend τ to $\hat{\tau}:\hat{\mathbb{E}}^n\to S^n$ where $\hat{\tau}|\mathbb{E}^n=\tau$ and $\hat{\tau}(\infty)=-e_{n+1}$.

$\hat{\tau}:\hat{\mathbb{E}}^n\to S^n$ is a bijection which is clearly continuous at every point of $\mathbb{E}^n$.

Observe that

$$(d(\tau(x), -e_{n+1}))^2 = \|\tau(x) + e_{n+1}\|^2 = \left\|\frac{2}{1+\|x\|^2}(\bar{x} + e_{n+1})\right\|^2 =$$

$$\frac{4}{(1+\|x\|^2)^2}\|\bar{x} + e_{n+1}\|^2 = \frac{4}{(1+\|x\|^2)^2}(\|x\|^2+1) = \frac{4}{1+\|x\|^2}.$$

Thus, $\lim\limits_{\|x\| \to \infty} d(\tau(x), -e_{n+1}) = 0$. Hence, $\hat{\tau}$ is continuous at ∞. Thus $\hat{\tau}: \hat{E}^n \to S^n$ is a homeomorphism.

Observe that $\hat{\tau}(\hat{P}(u,a)) = S^n \cap Q$ where Q is the n-dimensional hyperplane in E^{n+1} that contains $(P(u,a) \times \{0\}) \cup \{-e_{n+1}\}$. Clearly $S^n \cap Q$ is homeomorphic to S^{n-1}. Hence, $\hat{P}(u,a)$ is homeomorphic to S^{n-1}.

[Here is more detail. Let $v = \frac{\bar{u} - a e_{n+1}}{\sqrt{1+a^2}}$ and $b = \frac{a}{\sqrt{1+a^2}}$. Let $Q = P(v, b)$.

Then $(P(u,a) \times \{0\}) \cup \{-e_{n+1}\} \subset Q$ and $\hat{\tau}(\hat{P}(u,a)) = S^n \cap Q$. (Verify these statements.) Let $c = \frac{a}{1+a^2}(\bar{u} - a e_{n+1})$ and $r = \frac{1}{\sqrt{1+a^2}}$. Then $S^n \cap Q = S(c,r) \cap Q$ and

$c \in Q$. Since $c \in Q$, then $S(cr) \cap Q$ is an $(n-1)$-sphere in Q of radius r centered at c. Since $\hat{\tau}(\hat{P}(ua)) = S(cr) \cap Q$, then $\hat{\tau}(\hat{P}(ua))$ is homeomorphic to S^{n-1}.]

Proof of b) If C is a compact subset of E^n, then so is $Z_{ua}(C)$. Since $\hat{Z}_{ua}(\hat{E}^n - Z_{ua}(C)) = \hat{E}^n - C$, then $\hat{Z}_{ua}$ is continuous at ∞. Thus $\hat{Z}_{ua}$ is a homeomorphism.

To prove the continuity of $\hat{I}_{cr}$ at c, let C be a compact subset of E^n. There is an $s > 0$ such that $C \subset B(c,s)$, where $B(c,s) = \{x \in E^n : \|x-c\| \leq s\}$. Since $\| I_{cr}(x) - c\| = r^2/\|x-c\|$, then

$$\|x-c\| < r^2/s \Rightarrow \| I_{cr}(x) - c\| > r^2/(r^2/s) = s.$$

Thus, $\hat{I}_{cr}(B(c, r^2/s)) \subset \hat{E}^n - B(c,s) \subset \hat{E}^n - C$.

Hence, $\hat{I}_{cr}$ is continuous at c.

To prove the continuity of $\hat{I}_{cr}$ at ∞, let $s > 0$. Then

$\|x-c\| > r^2/s \Rightarrow \|I_{cr}(x) - c\| < r^2/(r^2/s) = s$.

Hence, $\hat{I}_{cr}(\hat{\mathbb{E}}^n - B(c, r^2/s)) \subset B(c,s)$.
Thus, $\hat{I}_{cr}$ is continuous at ∞.

The proofs of c) and d) are easy. □

We can now give an accurate statement of:

<u>A Theorem of Liouville</u>. For $n \geq 3$ every conformal diffeomorphism between open subsets of $\mathbb{E}^n$ is a finite composition of extended reflections and extended inversions

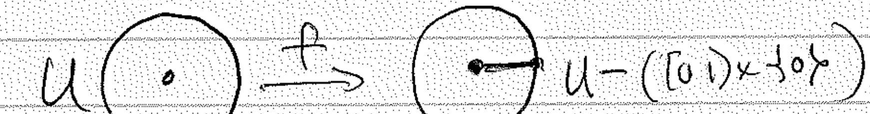
For a proof see R. Benedetti and C. Petronio, <u>Lectures on Hyperbolic Geometry</u>, p 15

In particular, we are now able to write a correct equation that expresses the dilation $D_{c,a}$ as the composition of two extended inversions:

$$D_{ca} = \hat{I}_{cs} \circ \hat{I}_{cr} \mid \mathbb{E}^n$$

whenever $s/r = \sqrt{a}$.

When $n=2$, the statement of Liouville's Theorem is false. Indeed, the Riemann Mapping Theorem implies that every simply connected proper open subset of $\mathbb{E}^2$ is the image of the open unit disk $U^2 = \{x \in \mathbb{E}^2 : \|x\| < 1\}$ under a conformal diffeomorphism.
However, an open subset of $\mathbb{E}^2$ which is the image of U^2 under a finite composition of extended reflections and extended inversions must be either an open disk of the form $\{x \in \mathbb{E}^2 : \|x - c\| < r\}$ for some $c \in \mathbb{E}^2$ and $r > 0$ or an open half space of the form $\{x \in \mathbb{E}^2 : x \circ u > a\}$ where u is a unit vector in $\mathbb{E}^2$ and $a \in \mathbb{R}$. Since there are simply connected proper open subsets of $\mathbb{E}^2$ that are not open disks or open half spaces, then there are conformal diffeomorphisms between open subsets of $\mathbb{E}^2$ that can't be expressed as finite compositions of extended reflections and extended inversions. For example, $U^2 - ([0,1) \times \{0\})$ is a simply connected proper open subset of $\mathbb{E}^2$. The Riemann Mapping Theorem provides a conformal diffeomorphism $f : U^2 \to U^2 - ([0,1) \times \{0\})$. Since $U^2 - ([0,1) \times \{0\})$ is not an open disk or open half space, then f can't be expressed as a finite composition of extended reflections and extended inversions



<u>Def</u> A homeomorphism of $\hat{\mathbb{E}}^n$ which is a finite composition of extended reflections and extended inversions is called a <u>Mobius transformation</u> of $\hat{\mathbb{E}}^n$. The set of all Mobius transformations of $\hat{\mathbb{E}}^n$ is clearly a group with respect to composition; it is called the <u>Mobius group of</u> $\hat{\mathbb{E}}^n$ and is denoted $\mathrm{Mob}(\hat{\mathbb{E}}^n)$. If $A \subset \hat{\mathbb{E}}^n$, let

$$\mathrm{Mob}(A) = \{\varphi \in \mathrm{Mob}(\hat{\mathbb{E}}^n) : \varphi(A) = A\}.$$

Clearly, $\mathrm{Mob}(A)$ is a subgroup of $\mathrm{Mob}(\hat{\mathbb{E}}^n)$. $\mathrm{Mob}(A)$ is called the <u>Mobius group of</u> A

<u>Notation</u> Let $B^n = \{x \in \mathbb{E}^n : \|x\| \le 1\}$, $U^n = \{x \in \mathbb{E}^n : \|x\| < 1\}$, $\mathbb{E}^n_+ = \{x \in \mathbb{E}^n : x \cdot e_n > 0\}$ and $\mathbb{E}^n_- = \{x \in \mathbb{E}^n : x \cdot e_n < 0\}$.

We will study the Mobius groups of $\hat{\mathbb{E}}^n$, U^n and $\mathbb{E}^n_+$ further.

<u>Def</u> Let Σ^n denote ~~the~~ the set of all extended hyperplanes and hyperspheres in $\hat{\mathbb{E}}^n$. If $S \in \Sigma^n$, let J_S denoted the extended reflection / inversion in S. Thus:

$$J_S = \begin{cases} \hat{Z}_{ua} & \text{if } S = \hat{P}(u,a) \\ \hat{I}_{cr} & \text{if } S = S(cr) \end{cases}$$

Clearly $\{J_S : S \in \Sigma^n\}$ generates $\mathrm{Mob}(\hat{\mathbb{E}}^n)$.

Theorem 4.5 $Mob(\hat{E}^n)$ acts transitively on Σ^n. In other words:
a) For each $\phi \in Mob(\hat{E}^n)$ and $S \in \Sigma^n$, $\phi(S) \in \Sigma^n$, and
b) For any two $S, T \in \Sigma^n$, there is a $\phi \in Mob(\hat{E}^n)$ such that $\phi(S) = T$.

Proof of a) It suffices to prove that for all $S, T \in \Sigma^n$, $J_T(S) \in \Sigma^n$.

First consider the case $T = \hat{P}(v,b)$ and $J_T = \hat{Z}_{vb}$ and $S = \hat{P}(u,a)$. Let $w = \hat{Z}_{v0}(u)$ and $c = a + 2b(v \cdot w)$. Then $\hat{Z}_{vb}(\hat{P}(ua)) = \hat{P}(w,c)$.

[Proof Let $x \in \dot{P}(ua)$. Then $x \cdot u = a$. Note that
$Z_{vb}(x) = x - 2(x \cdot v - b)v = x - 2(x \cdot v)v + 2bv =$
$Z_{v0}(x) + 2bv$. Hence,

$Z_{vb}(x) \cdot w = Z_{v0}(x) \cdot w + 2b(v \cdot w) = Z_{v0}(x) \cdot Z_{v0}(u) + 2b(v \cdot w) =$
$x \cdot u + 2b(v \cdot w) = a + 2b(v \cdot w) = c.$

Thus, $Z_{vb}(x) \in P(w,c)$. This proves
$\hat{Z}_{vb}(\hat{P}(ua)) \subset \hat{P}(w,c)$.

Now let $u' = Z_{u0}(w)$ and $a' = c + 2b(v \cdot u')$.
Then the preceding argument shows
$\hat{Z}_{vb}(\hat{P}(w,c)) \subset \hat{P}(u', a')$. Now observe

that $u' = Z_{u0}(w) = Z_{u0} \circ Z_{u0}(u) = u$. Also

$a' = c + 2b(v \cdot u') = (a + 2b(v \cdot w)) + 2b(v \cdot u') =$

$a + 2b(v \cdot Z_{v0}(u)) + 2b(v \cdot u) =$

$a + 2b\, v \cdot (u - 2(u \cdot v)v) + 2b(v \cdot u) =$

$a + 2b(v \cdot u) - 4b(u \cdot v)\|v\|^2 + 2b(v \cdot u) =$

$a + 4b(v \cdot u) - 4b(u \cdot v) = a$.

Hence, $\hat{Z}_{vb}(\hat{P}(w,c)) \subset \hat{P}(u,a)$.

Apply $\hat{Z}_{vb}$ to this inclusion to obtain

$\hat{P}(w,c) \subset \hat{Z}_{vb}(\hat{P}(u,a))$.

We conclude that $\hat{Z}_{vb}(\hat{P}(u a)) = \hat{P}(w,c)$.]

Next consider the case $T = \hat{P}(v,b)$ and $J_T = \hat{Z}_{vb}$ and $S = S(c,r)$. Let $d = \hat{Z}_{vb}(c)$. Since Z_{vb} is an isometry, then clearly $\hat{Z}_{vb}(S(cr)) \subset S(d,r)$. Also since $\hat{Z}_{vb}(d) = \hat{Z}_{vb} \circ \hat{Z}_{vb}(c) = c$, then $\hat{Z}_{vb}(S(dr)) \subset S(cr)$. We apply $\hat{Z}_{vb}$ to the last inclusion to obtain $S(dr) \subset \hat{Z}_{vb}(S(cr))$. We conclude that $\hat{Z}_{vb}(S(cr)) = S(d,r)$.

Now suppose $T = S(cr)$ and $J_T = \hat{I}_{cr}$

We will reduce to considering $\hat{I}_{01}$.

Define $\lambda : \hat{\mathbb{E}}^n \to \hat{\mathbb{E}}^n$ by

$$\lambda(x) = \begin{cases} rx + c & \text{if } x \in \mathbb{E}^n \\ \infty & \text{if } x = \infty \end{cases}$$

Clearly λ is invertible and λ^{-1} obeys the equations

$$\lambda^{-1}(y) = \begin{cases} \frac{1}{r}(y-c) & \text{if } y \in \mathbb{E}^n \\ \infty & \text{if } y = \infty \end{cases}$$

Then $\hat{I}_{cr} \circ \lambda = \lambda \circ \hat{I}_{01}$

[Proof. For $x \in \mathbb{E}^n - \{0\}$, $\hat{I}_{cr} \circ \lambda(x) = I_{cr}(rx+c) =$

$\frac{r^2}{\|(rx+c)-c\|^2}((rx+c)-c) + c = \frac{r^2}{r^2\|x\|^2}(rx) + c =$

$r\left(\frac{x}{\|x\|^2}\right) + c = r\left(\hat{I}_{01}(x)\right) + c = \lambda \circ \hat{I}_{01}(x)$.

Also $\hat{I}_{cr} \circ \lambda(0) = \hat{I}_{cr}(c) = \infty = \lambda(\infty) = \lambda \circ I_{01}(0)$

and $\hat{I}_{cr} \circ \lambda(\infty) = \hat{I}_{cr}(\infty) = c = \lambda(0) = \lambda \circ \hat{I}_{01}(\infty)$.]

Hence, $\hat{I}_{cr} = \lambda \circ \hat{I}_{01} \circ \lambda^{-1}$.

Next observe that λ and λ^{-1} take

elements of Σ^n to elements of Σ^n.
Indeed $\lambda(\hat{P}(u,a)) = \hat{P}(u, ra + c\cdot u)$,
$\lambda^{-1}(\hat{P}(u,a)) = \hat{P}(u, \frac{1}{r}(a - c\cdot u))$,
$\lambda(S(d,s)) = S(\lambda(d), rs)$ and
$\lambda^{-1}(S(d,s)) = S(\lambda^{-1}d, s/r)$.
(Exercise: Verify these equations.)

Since $\hat{I}_{cr} = \lambda \circ \hat{I}_{01} \circ \lambda^{-1}$ and λ and λ^{-1} take elements of Σ^n to elements of Σ^n, then it suffices to prove $\hat{I}_{01}$ takes elements of Σ^n to elements of Σ^n.

Consider the equation
$$a\|x\|^2 + b\cdot x + c = 0 \quad \ldots (*)$$
subject to the constraints: if $a=0$, then $b \neq 0$, and if $a \neq 0$, then $\|b\|^2 - 4ac > 0$.
Hyperplanes and hyperspheres are solution sets of equations of this type.
Indeed, $x \in P(u,a) \iff u\cdot x - a = 0$ where $u \neq 0$.
Also $x \in S(c,r) \iff \|x-c\|^2 = r^2 \iff$
$\|x\|^2 - 2c\cdot x + (\|c\|^2 - r^2) = 0$ where
$\|-2c\|^2 - 4(1)(\|c\|^2 - r^2) = 4\|c\|^2 - 4\|c\|^2 + 4r^2 = 4r^2 > 0$

Conversely, the solution sets of (*) are hyperplanes and hyperspheres.

Indeed, if $a = 0$ and $\mathbb{b} \neq 0$, then:

x is a solution of (*) $\iff \mathbb{b}x = -c \iff$

$\frac{\mathbb{b}}{\|b\|} x = -\frac{c}{\|b\|} \iff x \in P\left(\frac{\mathbb{b}}{\|b\|}, -\frac{c}{\|b\|}\right)$.

Also if $a \neq 0$ and $\|b\|^2 - 4ac > 0$, then:

x is a solution of (*) $\iff a\|x\|^2 + \mathbb{b}\cdot x + c = 0 \iff$

$\|x\|^2 + \frac{\mathbb{b}}{a}\cdot x + \frac{c}{a} = 0 \iff \|x\|^2 + \frac{\mathbb{b}}{a}\cdot x + \left\|\frac{\mathbb{b}}{2a}\right\|^2 = \left\|\frac{b}{2a}\right\|^2 - \frac{c}{a}$

$\iff \left\|x - \frac{\mathbb{b}}{2a}\right\|^2 = \frac{\|b\|^2 - 4ac}{4a^2} \iff x \in S\left(\frac{\mathbb{b}}{2a}, \frac{\sqrt{\|b\|^2 - 4ac}}{2|a|}\right)$

Let $S \in \Sigma^n$. Then $S - \{\infty\}$ is the solution set of an equation $a\|x\|^2 + \mathbb{b}\cdot x + c = 0$ where either $a = 0$ and $\mathbb{b} \neq 0$, or $a \neq 0$ and $\|b\|^2 - 4ac > 0$. Let T be the solution set of the equation $c\|y\|^2 + \mathbb{b}\cdot y + a = 0$. together with the point ∞ if this solution set is non-compact.

If $c = 0$, then $a = 0 \Rightarrow \|b\| \neq 0$, and

$a \neq 0 \Rightarrow \|b\|^2 \cancel{=} \|b\|^2 - 4ac > 0 \Rightarrow \mathbb{b} \neq 0$.

If $c \neq 0$, then $a = 0 \Rightarrow \|b\| \neq 0 \Rightarrow$
$\|b\|^2 - 4ca = \|b\|^2 > 0$, and if $a \neq 0$,
then $\|b\|^2 - 4ca = \|b\|^2 - 4ac > 0$.

Thus, the equation $c\|y\|^2 + \mathbb{b}\cdot y + a = 0$ obeys the constraints that imply $T \in \Sigma$.

We now show $\hat{I}_{01}(S) \subset T$. Let $x \in S - \{0, \infty\}$ and let $y = I_{01}(x)$. Then $x = I_{01}(y) = \frac{y}{\|y\|^2}$

Hence $\frac{y}{\|y\|^2} \in S$. Thus,

$$a \left\| \frac{y}{\|y\|^2} \right\|^2 + b \cdot \left(\frac{y}{\|y\|^2} \right) + c = 0$$

Therefore, $\frac{a}{\|y\|^2} + \frac{b \cdot y}{\|y\|^2} + c = 0$.

Hence, $c\|y\|^2 + b \cdot y + a = 0$.

Thus, $y \in T$. This proves $I_{01}(S - \{0, \infty\}) \subset T$. Let $\overline{S - \{0, \infty\}}$ denote the closure of $S - \{0, \infty\}$ in $\hat{E}^n$. Clearly $\overline{S - \{0, \infty\}} = S$. Since $\hat{I}_{01}$ is continuous and T is a closed subset of $\hat{E}^n$, then $\hat{I}_{01}(S) = \hat{I}_{01}(\overline{S - \{0, \infty\}}) \subset T$.

If we interchange the roles of a and c, the preceding argument shows $\hat{I}_{01}(T) \subset S$. Applying $\hat{I}_{01}$ to both sides of this inclusion, we get $T \subset \hat{I}_{01}(S)$. Thus, $\hat{I}_{01}(S) = T$.

We have shown that $\hat{I}_{01}$ takes elements of Σ^n to elements of Σ^n. It follows that every extended inversion $\hat{I}_{cr}$ takes elements of Σ^n to elements of Σ^n. $\square$

Proof of b) Let

$\Sigma^n_P = \{S \in \Sigma^n : S \text{ is an extended hyperplane}\}$ and

$\Sigma^n_S = \{S \in \Sigma^n : S \text{ is a hypersphere}\}$. Then

$\Sigma^n = \Sigma^n_P \cup \Sigma^n_S$ and $\Sigma^n_P \cap \Sigma^n_S = \phi$.

First we show that if $S, T \in \Sigma^n_P$, then there is a $\phi \in Mob(\hat{E}^n)$ such that $\phi(S) = T$. Let $S = \hat{P}(ua)$ and $T = \hat{P}(vb)$.

Assume $u \neq v$. Let $w = \frac{v-u}{\|v-u\|}$ and

$c = \frac{b-a}{2(w \cdot v)}$. $\left(w \cdot v = \frac{(v-u)\cdot v}{\|v-u\|} = \frac{1 - u \cdot v}{\|v-u\|} \neq 0 \text{ because } u \neq v. \right)$

Then $\hat{Z}_{wc}(\hat{P}(ua)) = \hat{P}(v, b)$.

Now assume $u = v$. Let $c = \frac{a+b}{2}$.

Then $\hat{Z}_{u,c}(\hat{P}(ua)) = \hat{P}(vb)$.

Second consider $S, T = \Sigma^n_S$.

Say $S = S(c,r)$ and $T = S(d,s)$

If $c = d$, let $\phi = id_{\hat{\mathbb{E}}^n}$. Then $\phi \in Mob(\hat{\mathbb{E}}^n)$ and $\phi(S(c,r)) = S(d,r)$.

If $c \neq d$, let $u = \frac{d-c}{\|d-c\|}$, $m = \frac{c+d}{2}$ and $a = m \cdot u$. Let $\phi = \hat{\rho}_{ua}$. Then $\phi \in Mob(\hat{\mathbb{E}}^n)$, $\phi(c) = d$ and $\phi(S(c,r)) = S(d,r)$.

If $s = r$, we're done; so assume $r \neq s$.

Recall $\hat{D}_{d,s/r} : \hat{\mathbb{E}}^n \to \hat{\mathbb{E}}^n$ is defined by

$$\hat{D}_{d,s/r}(x) = \begin{cases} s/r\,(x-d) + d & \text{if } x \in \mathbb{E}^n \\ \infty & \text{if } x = \infty \end{cases}$$

Clearly $\hat{D}_{d,s/r}(S(d,r)) = S(d,s)$.

Recall that $\hat{D}_{d,s/r} = \hat{I}_{d,\sqrt{s}} \circ \hat{I}_{d,\sqrt{r}}$.

Hence $\hat{D}_{d,s/r} \in Mob(\hat{\mathbb{E}}^n)$. Thus $\hat{D}_{d,s/r} \circ \phi \in Mob(\hat{\mathbb{E}}^n)$ and $\hat{D}_{d,s/r} \circ \psi(S(c,r)) = S(d,s)$.

To complete the proof of b), it suffices to prove that for some $S_0 \in \Sigma_P^n$ and $T_0 \in \Sigma_S^n$, there is a $\phi \in Mob(\hat{\mathbb{E}}^n)$ such that $\phi(S_0) = T_0$.

Let $S_0 = \hat{\mathbb{E}}^{n-1} = \hat{P}(e_n, 0)$ and let $T_0 = S^{n-1} = S(0,1)$. Let $\phi = \hat{I}_{-e_n, \sqrt{2}}$.

Then $\phi \in Mob(\hat{\mathbb{E}}^n)$. We will prove $\phi(S_0) = T_0$.

$\infty \in S_0$ and $\phi(\infty) = -e_n \in T_0$.

Let $x \in S_0 - \{\infty\} = \mathbb{E}^{n-1} \times \{0\}$. Then

$$\phi(x) = \frac{2}{\|x+e_n\|^2}(x+e_n) - e_n.$$

Since $x \cdot e_n = 0$, then $\|x+e_n\|^2 = \|x\|^2 + 1$.

Hence $\phi(x) = \frac{2x}{1+\|x\|^2} + \left(\frac{2}{1+\|x\|^2} - 1\right)e_n =$

$\frac{2x}{1+\|x\|^2} + \left(\frac{1-\|x\|^2}{1+\|x\|^2}\right)e_{n+1}$. Therefore,

$$\|\phi(x)\|^2 = \frac{4\|x\|^2}{(1+\|x\|^2)^2} + \left(\frac{1-\|x\|^2}{1+\|x\|^2}\right)^2 =$$

$$\frac{4\|x\|^2 + (1-2\|x\|^2+\|x\|^4)}{(1+\|x\|^2)^2} = \frac{1+2\|x\|^2+\|x\|^4}{(1+\|x\|^2)} = \frac{(1+\|x\|^2)^2}{(1+\|x\|^2)^2} = 1.$$

Hence, $\phi(S_0 - \{\infty\}) \subset S^{n-1} = T_0$. So $\phi(S_0) \subset T_0$.

Next we prove $\phi(T_0) \subset S_0$.

First $\phi(-e_n) = \infty \in S_0$, Now suppose $x \in T_0 - \{-e_n\}$. Then $\|x\| = 1$.

$$\phi(x) = \frac{2}{\|x+e_n\|^2}(x+e_n) - e_n .$$

$$\|x+e_n\|^2 = \|x\|^2 + 2x \cdot e_n + \|e_n\|^2 = 2(1+x_n)$$

Thus $\phi(x) = \frac{x}{1+x_n} + \left(\frac{1}{1+x_n} - 1\right)e_n =$

$\frac{x}{1+x_n} - \frac{x_n}{1+x_n} e_n$. Therefore

$$\phi(x) \cdot e_n = \frac{x_n}{1+x_n} - \frac{x_n}{1+x_n} = 0 .$$

So $\phi(x) \in \mathbb{R}^{n-1} \times \{0\} \subset S_0$.

Hence, $\phi(T_0 - \{-e_n\}) \subset S_0$.

Thus $\phi(T_0) \subset S_0$.

It follows that $T_0 = \phi \circ \phi(T_0) \subset \phi(S_0)$,

We conclude that $\phi(S_0) = T_0$. $\square$

Homework Problem 4.1.

a) Verify the following assertions made in the proof of Theorem 4.5.6. Let $\hat{P}(u,a)$ and $\hat{P}(v,b)$ be extended hyperplanes in $\hat{\mathbb{E}}^n$.

- If $u \neq v$, $w = \frac{v-u}{\|v-u\|}$ and $c = \frac{b-a}{2(w \cdot v)}$, then $\hat{Z}_{wc}(\hat{P}(ua)) = \hat{P}(v,b)$.
- If $u = v$ and $c = \frac{a+b}{2}$, then $\hat{Z}_{u,c}(\hat{P}(ua)) = \hat{P}(v,b)$.

b). Let $S(c,r)$ and $S(d,s)$ be hyperspheres in $\hat{\mathbb{E}}^n$. If $r \neq s$, does there exist a single extended inversion $\hat{I}_{e,t}$ such that $\hat{I}_{e,t}(S(c,r)) = S(d,s)$?

<u>Def.</u> If S is a k-dimensional vector subspace of an n-dimensional inner product space V, let

$$S^{\perp} = \{x \in V : x \cdot y = 0 \text{ for every } y \in S\}.$$

Then $S^{\perp}$ is an $(n-k)$-dimensional vector subspace of V called the <u>orthogonal complement of S</u>. If S and T are $(n-1)$-dimensional vector subspaces of an n-dimensional inner product space V, then S <u>is orthogonal to T</u> if any one of the following three equivalent conditions holds:

- $S^{\perp} \subset T$.
- $T^{\perp} \subset S$.
- For all $x \in S^{\perp}$ and $y \in T^{\perp}$, $x \cdot y = 0$.

<u>Lemma 4.6</u> If S and T are $(n-1)$-dimensional vector subspaces of an n-dimensional inner product space V, then the following conditions are equivalent.

a) $S^{\perp} \subset T$.
b) $T^{\perp} \subset S$
c) For all $x \in S^{\perp}$ and $y \in T^{\perp}$, $x \cdot y = 0$.

<u>Proof that a) implies c)</u> Assume $S^{\perp} \subset T$. Let $x \in S^{\perp}$ and $y \in T^{\perp}$. Then $x \in T$. Hence $x \cdot y = 0$. This proves c). $\square$

Proof that c) implies a). Assume $x \cdot y = 0$ for all $x \in S^\perp$ and $y \in T^\perp$. There is an orthonormal basis $u_1, u_2, \dots, u_n$ for V such that $u_1, u_2, \dots, u_{n-1} \in T$ and $u_n \in T^\perp$. Let $x \in S^\perp$. Then $x = \sum_{i=1}^{n} (x \cdot u_i) u_i$. By hypothesis $x \cdot u_n = 0$. Hence, $x = \sum_{i=1}^{n-1} (x \cdot u_i) u_i$. Therefore, $x \in T$. This proves $S^\perp \subset T$. □

The proof that b) is equivalent to c) is similar. □

Def Suppose L and M are $(n-1)$-dimensional differentiable submanifolds of an n-dimensional Riemannian manifold N. For $x \in L \cap M$, L is orthogonal to M at x if $T_x(L)$ is orthogonal to $T_x(M)$ in $T_x(N)$. L is orthogonal to M if $L \cap M \neq \phi$ and L is orthogonal to M at every point of $L \cap M$.

Lemma 4.7. Suppose L and M are $(n-1)$-dimensional differentiable submanifolds of an n-dimensional Riemannian manifold N, and suppose $\varphi : N \to N'$ is a conformal diffeomorphism from N to an n-dimensional Riemannian manifold N'. If L is orthogonal to M at a point $x \in L \cap M$, then $\varphi(L)$ is orthogonal to $\varphi(M)$ at $\varphi(x)$. Hence, if L is orthogonal to M, then $\varphi(L)$ is orthogonal to $\varphi(M)$.

<u>Proof</u> Assume L is orthogonal to M at the point $x \in L \cap M$. Then $T_x(L)$ is orthogonal to $T_x(M)$ in $T_x(N)$. We must prove $T_{\varphi(x)}(\varphi(L))$ is orthogonal to $T_{\varphi(x)}(\varphi(M))$ in $T_{\varphi(x)}(N')$. Since $\varphi(N) = N'$, then $T_{\varphi(x)}(N') = d\varphi_x(T_x(N))$. Also $T_{\varphi(x)}(L) = d\varphi_x(T_x(L))$ and $T_{\varphi(x)}(M) = d\varphi_x(T_x(M))$

We claim that if S is a vector subspace of $T_x(N)$, then $d\varphi_x(S^{\perp}) = (d\varphi_x(S))^{\perp}$.
To prove this let $v \in d\varphi_x(S^{\perp})$ and let $w \in d\varphi_x(S)$
Then $v = d\varphi_x(v')$ and $w = d\varphi_x(w')$ where $v \in S^{\perp}$ $w \in S$. Hence, $v' \cdot w' = 0$. Since φ is conformal, then $d\varphi_x$ is angle preserving. Hence, there is an $r > 0$ such that $d\varphi_x(v) \cdot d\varphi_x(w) = r^2(v' \cdot w')$. (by Lemma 4.1). Hence $v \cdot w = r^2(0) = 0$. This proves $v \in d\varphi_x(S)$. Thus, $d\varphi_x(S^{\perp}) \subset (d\varphi_x(S))^{\perp}$.
To prove the opposite inclusion, let $v \in (d\varphi_x(S))^{\perp}$.
There is a $v' \in T_x(N)$ such that $d\varphi_x(v') = v$.
Let $w' \in S$. Then $d\varphi_x(w') \in d\varphi_x(S)$.

Hence $0 = \cancel{v} \circ d\varphi_x(w') = d\varphi_x(v') \circ d\varphi_x(w')$ $= r^2(v' \circ w')$. Thus $v' \circ w' = 0$. This proves $v' \in S^\perp$. Hence $v = d\varphi_x(v') \in d\varphi_x(S^\perp)$. We have proved $(d\varphi_x(S))^\perp \subset d\varphi_x(S^\perp)$.

We conclude that $d\varphi_x(S^\perp) = (d\varphi_x(S))^\perp$.

Since $T_x(L)$ is orthogonal to $T_x(M)$, then $(T_x(L))^\perp \subset T_x(M)$. Hence, $d\varphi_x((T_x(L))^\perp) \subset d\varphi_x(T_x(M))$. Since $d\varphi_x((T_x(L))^\perp) = (d\varphi_x(T_x(L)))^\perp = (T_{\varphi(x)}(\varphi(L)))^\perp$ and $d\varphi_x(T_x(M)) = T_{\varphi(x)}(\varphi(M))$, then $(T_{\varphi(x)}(\varphi(L)))^\perp \subset T_{\varphi(x)}(\varphi(M))$. Hence, $T_{\varphi(x)}(\varphi(L))$ is orthogonal to $T_{\varphi(x)}(\varphi(M))$. We conclude that $\varphi(L)$ is orthogonal to $\varphi(M)$ at $\varphi(x)$. $\square$

<u>Theorem 4-8.</u> a) The following statements are equivalent:

i) $P(u,a)$ and $P(v,b)$ are orthogonal at some point of $P(u,a) \cap P(v,b)$

ii) $P(u,a)$ and $P(v,b)$ are orthogonal.

iii) $u \cdot v = 0$.

b) The following statements are equivalent:

i) $P(u,a)$ and $S(c,r)$ are orthogonal at some point of $P(u,a) \cap S(c,r)$.

ii) $P(u,a)$ and $S(c,r)$ are orthogonal

iii) $c \in P(u,a)$.

c) The following statements are equivalent

i) $S(c,r)$ and $S(d,s)$ are orthogonal at some point of $S(c,r) \cap S(d,s)$.

ii) $S(c,r)$ and $S(d,s)$ are orthogonal

iii) $r^2+s^2 = \|c-d\|^2$.

Proof of a) We prove i) implies iii).
Assume $P(ua)$ and $P(vb)$ are orthogonal at $x \in P(ua) \cap P(vb)$.

We claim $T_x(P(ua)) = P(u0)$.

First let $w \in T_x(P(ua))$. Then there is a differential path $\gamma : (-\varepsilon, \varepsilon) \to P(ua)$ such that $\gamma(0) = x$ and $\gamma'(0) = w$. Since $\gamma(-\varepsilon, \varepsilon) \subset P(ua)$, then $\gamma(t) \cdot u = a$ for all $t \in (-\varepsilon, \varepsilon)$. Hence $w \cdot u = \gamma'(0) \cdot u = 0$. Thus $w \in P(u, 0)$. This proves $T_x(P(ua)) \subset P(u0)$.

For the converse, let $w \in P(u0)$. Define $\gamma : \mathbb{R} \to \mathbb{E}^n$ by $\gamma(t) = tw + x$. Then
$\gamma(t) \cdot u = t(w \cdot u) + x \cdot u = t \cdot 0 + a = a$.
Thus $\gamma : \mathbb{R} \to P(u, a)$. Clearly $\gamma(0) = x$.
Thus, $w = \gamma'(0) \in T_x(P(ua))$. This proves $P(u0) \subset T_x(P(ua))$. We conclude that $T_x(P(ua)) = P(u0)$.

Similarly $T_x(P(v, b)) = P(v0)$.

For each $w \in P(u, 0)$, $u \cdot w = 0$.
Thus, $u \in P(u, 0)^{\perp}$. Similarly $v \in P(v0)^{\perp}$.

Thus, $u \in T_x(P(ua))^{\perp}$ and $v \in T_x(P(vb))^{\perp}$.
Since $P(ua)$ and $P(vb)$ are orthogonal at x, we conclude that $u \cdot v = 0$.

Next we prove iii) implies ii).
Assume $u \cdot v = 0$. Suppose $x \in P(ua) \cap P(vb)$.
As we showed in the previous paragraph, $T_x(P(ua)) = P(u0)$, $T_x(P(vb)) = P(v0)$, $u \in P(u0)^{\perp}$ and $v \in P(vb)^{\perp}$.
Since $P(u0)$ and $P(v0)$ are $(n-1)$-dimensional, then $P(u0)^{\perp}$ and $P(v0)^{\perp}$ are 1-dimensional.
Since $u \neq 0$ and $v \neq 0$, it follows that $P(u0)^{\perp} = \{tu : t \in \mathbb{R}\}$ and $P(v0)^{\perp} = \{tv : t \in \mathbb{R}\}$.
It follows that if $u' \in T_x(P(ua))$ and $v' \in T_x(P(vb))$, then $u' = su$ and $v' = tv$ for some $s, t \in \mathbb{R}$. Therefore $u' \cdot v' = st(u \cdot v) = 0$. This shows $P(ua)$ and $P(vb)$ are orthogonal at x. We have proved ii).

Clearly ii) implies i). $\square$

Proof of 6) We prove i) implies ii).
Assume $P(ua)$ and $S(cr)$ are orthogonal at $x \in P(ua) \cap S(cr)$.
We showed previously that $T_x(P(ua)) = P(u0)$ and, hence, $u \in T_x(P(ua))^{\perp}$. We will now show that $T_x(S(cr)) = P(x-c, 0)$ and, hence, that $x-c \in T_x(S(cr))^{\perp}$. Let $v \in T_x(S(cr))$.

Then there is a differentiable curve $\gamma: (-\varepsilon, \varepsilon) \to S(cr)$ such that $\gamma(0) = x$ and $\gamma'(0) = v$. Hence, $\|\gamma(t) - c\|^2 = r^2$ for all $t \in (-\varepsilon, \varepsilon)$. Differentiating this equation yields $2(\gamma(t) - c) \cdot \gamma'(t) = 0$. Set $t = 0$. Then $(x-c) \cdot v = 0$. Thus $v \in P(\frac{x-c}{\|x-c\|}, 0)$.
Consequently, $T_x(S(cr)) \subset P(\frac{x-c}{\|x-c\|}, 0)$.
Now let $v \in P(\frac{x-c}{\|x-c\|}, 0)$. Define $\gamma: \mathbb{R} \to E^n$ by

$$\gamma(t) = \cos\left(\frac{\|v\|}{r} t\right)(x-c) + \sin\left(\frac{\|v\|}{r} t\right) \frac{rv}{\|v\|} + c,$$

Since $(x-c) \cdot v = 0$, and $\|x-c\|^2 = r^2$, then

$$\|\gamma(t) - c\|^2 = \cos^2\left(\frac{\|v\|}{r} t\right)\|x-c\|^2 + \sin^2\left(\frac{\|v\|}{r} t\right) r^2 \frac{\|v\|^2}{\|v\|^2} = r^2.$$

Therefore, $\gamma: \mathbb{R} \to S(c, r)$. Clearly $\gamma(0) = x$.

$$\gamma'(t) = -\frac{\|v\|}{r} \sin\left(\frac{\|v\|}{r} t\right)(x-c) + \frac{\|v\|}{r} \cos\left(\frac{\|v\|}{r} t\right) \frac{rv}{\|v\|}.$$

Thus $\gamma'(0) = \frac{\|v\|}{r} \frac{r}{\|v\|} v = v$. It follows that $v \in T_x(S(cr))$. Thus $P(\frac{x-c}{\|x-c\|}, 0) \subset T_x(S(cr))$.
We conclude that $T_x(S(cr)) = P(\frac{x-c}{\|x-c\|}, 0)$.

Thus, $\cancel{x}-c \in T_x(S(cr))^{\perp}$.

Since $P(ua)$ and $S(cr)$ are orthogonal at x, and $u \in T_x(P(ua))^{\perp}$ and $x-c \in T_x(S(cr))^{\perp}$, then it follows that $(x-c)\cdot u = 0$. Thus $c\cdot u = x\cdot u = a$. Therefore $c \in P(ua)$.

Now we prove (iii) implies (ii). Assume $c \in P(ua)$. Suppose $x \in P(ua) \cap S(cr)$. Assume we argued above $u \in T_x(P(ua))^{\perp}$ and $x-c \in T_x(S(cr))^{\perp}$. Since $T_x(P(ua)) = P(u0)$ and $T_x(S(cr)) = P(\frac{x-c}{\|x-c\|}, 0)$ are $(n-1)$-dimensional, then $T_x(P(ua))^{\perp}$ and $T_x(S(cr))^{\perp}$ are 1-dimensional. Since $u \neq 0$ and $x-c \neq 0$, then it follows that $T_x(P(ua))^{\perp} = \{tu : t \in \mathbb{R}\}$ and $T_x(S(cr))^{\perp} = \{t(x-c) : t \in \mathbb{R}\}$. Let $u' \in T_x(P(ua))^{\perp}$ and $v' \in T_x(S(cr))^{\perp}$. Then $u' = su$ and $v' = t(x-c)$ for some $s, t \in \mathbb{R}$. Hence, $u'\cdot v' = (st)u\cdot(x-c) = (st)(x\cdot u - c\cdot u) = st(a-a) = 0$ because $x, c \in P(ua)$. This proves $P(ua)$ and $S(cr)$ are orthogonal at x, proving ii).

Clearly ii) implies i).

Proof of c) We prove i) implies iii).

Assume $S(cr)$ and $S(ds)$ are orthogonal at $x \in S(cr) \cap S(ds)$. We showed above that $x-c \in T_x(S(cr))^{\perp}$ and $x-d \in T_x(S(ds))^{\perp}$. Hence $(x-c)\cdot(x-d) = 0$. Therefore,

$$\|c-d\|^2 = \|(x-c)-(x-d)\|^2 =$$

$$\|x-c\|^2 - 2(x-c)\cdot(x-d) + \|x-d\|^2 =$$

$$r^2 - 2\cdot 0 + s^2 = r^2+s^2.$$

This proves iii).

Next we prove iii) implies ii). Assume $\|c-d\|^2 = r^2+s^2$. Therefore,

$\|(x-c)-(x-d)\|^2 = \|x-c\|^2 + \|x-d\|^2$. Hence

$$\|x-c\|^2 - 2(x-c)\cdot(x-d) + \|x-d\|^2 = \|x-c\|^2 + \|x-d\|^2.$$

Therefore, $(x-c)\cdot(x-d) = 0$. We show above that $x-c \in T_x(S(cr))^{\perp}$ and $x-d \in T_x(S(d,s))^{\perp}$. Also $T_x(S(cr))^{\perp}$ and $T_x(S(ds))^{\perp}$ are 1-dimensional.

It follows that $T_x(S(cr))^{\perp} = \{t(x-c) : t \in \mathbb{R}\}$ and $T_x(S(ds))^{\perp} = \{t(x-d) : t \in \mathbb{R}\}$.

Let $u' \in T_x(S(cr))^{\perp}$ and $v' \in T_x(S(ds))^{\perp}$. Then $u' = s(x-c)$ and $v' = t(x-d)$ for some $s, t \in \mathbb{R}$. Hence,

$$u' \cdot v' = st(x-c) \cdot (x-d) = 0.$$

This proves $S(cr)$ and $S(ds)$ are orthogonal at x. We have proved ii).

Clearly iii) implies i). $\square$

<u>Theorem 4.9</u> For every $S \in \Sigma^n$, $J_S \in Mob(\mathbb{E}^n_+)$ if and only if $S - \{\infty\}$ is orthogonal to $\mathbb{E}^{n-1} \times \{0\}$.

<u>Proof</u> Assume $S - \{\infty\}$ is orthogonal to $\mathbb{E}^{n-1} \times \{0\}$.

First consider the case $S = P(u, a)$. Since $\mathbb{E}^{n-1} \times \{0\} = P(e_{n+1}, 0)$, then Theorem 4.8.a implies $u \cdot e_n = 0$. Hence, if $x \in \mathbb{E}^n_+$, then

$$J_S(x) \cdot e_n = Z_{ua}(x) \cdot e_n = (x - 2(x \cdot u - a)u) \cdot e_n = x \cdot e_n > 0.$$

So $J_S(x) \in \mathbb{E}^n_+$. This proves $J_S(\mathbb{E}^n_+) \subset \mathbb{E}^n_+$. Hence $\mathbb{E}^n_+ = J_S \circ J_S(\mathbb{E}^n_+) \subset J_S(\mathbb{E}^n_+)$. Thus, $J_S(\mathbb{E}^n_+) = \mathbb{E}^n_+$.

Second suppose $S = S(c, r)$. Then Theorem 4.8.b implies $c \in \mathbb{E}^{n-1} \times 0$. If $x \in \mathbb{E}^n_+$, then

$$J_S(x) \cdot e_n = I_{cr}(x) \cdot e_n = \left(\frac{r^2}{\|x - c\|^2}(x - c) + c\right) \cdot e_n = \frac{r^2}{\|x - c\|^2} x \cdot e_n > 0.$$

So $J_S(x) \in \mathbb{E}^n_+$. Thus, $J_S(\mathbb{E}^n_+) \subset \mathbb{E}^n_+$. Hence, as above, $J_S(\mathbb{E}^n_+) = \mathbb{E}^n_+$.

Hence, if $S - \{\infty\}$ is orthogonal to $\mathbb{E}^{n-1} \times 0$, then $J_S \in Mob(\mathbb{E}^n_+)$.

Now assume $S - \{\infty\}$ is not orthogonal to $\mathbb{E}^{n-1} \times \{0\}$.

Consider the case $S = P(ua)$. Then Theorem 4.8.a implies $u \cdot e_n \neq 0$.

First suppose $a = 0$. Then $J_S(u) = Z_{u,0}(u) = -u$ and $J_S(-u) = J_S \circ J_S(u) = u$. Since $u \cdot e_n \neq 0$, then either $u \in \mathbb{E}^n_+$ and $J_S(u) = -u \notin \mathbb{E}^n_+$, or $-u \in \mathbb{E}^n_+$ and $J_S(-u) = u \notin \mathbb{E}^n_+$. Hence, $J_S(\mathbb{E}^n_+) \neq \mathbb{E}^n_+$. So $J_S \notin \mathrm{Mob}(\mathbb{E}^n_+)$.

Second suppose $a \neq 0$. Then $J_S(3au) = Z_{ua}(3au) = 3au - 2((3au) \cdot u - a)u = -au$ and $J_S(-au) = J_S \circ J_S(3au) = 3au$. Since $a \neq 0$ and $u \cdot e_n \neq 0$, then either $3au \in \mathbb{E}^n_+$ and $J_S(3au) = -au \notin \mathbb{E}^n_+$, or $-au \in \mathbb{E}^n_+$ and $J_S(-au) = 3au \notin \mathbb{E}^n_+$. Hence, $J_S(\mathbb{E}^n_+) \neq \mathbb{E}^n_+$. So $J_S \notin \mathrm{Mob}(\mathbb{E}^n_+)$

Finally consider the case $S = S(cr)$. Then Theorem 4.8, b implies $c \notin \mathbb{E}^{n-1} \times \{0\}$. Thus either $c \in \mathbb{E}^n_+$ or $c \in \mathbb{E}^n_-$.

First suppose $c \in \mathbb{E}^n_+$. Then $c \cdot e_n > 0$. Hence there is a real number t such that

$$0 < t < \min\left\{c \cdot e_n, \frac{r^3}{c \cdot e_n}\right\}$$

Let $x = c - te_n$. Then $x \cdot e_n = c \cdot e_n - t > 0$ because $t < c \cdot e_n$. Hence, $x \in E^n_+$.

$$J_S(x) = \hat{I}_{cr}(x) = \frac{r^2}{\|-te_n\|^2}(-te_n) + c = -\frac{r^2}{t}e_n + c.$$

Thus, $J_S(x) \cdot e_n = -\frac{r^2}{t} + c \cdot e_n < 0$ because $t < \frac{r^2}{c \cdot e_n}$. Hence, $J_S(x) \notin E^n_+$.

Therefore $J_S(E^n_+) \neq E^n_+$. So $J_S \notin \mathrm{Mob}(E^n_+)$.

Second suppose $c \in E^n_-$. Then $c \cdot e_n < 0$. Hence there is a real number t such that

$$t > \max\left\{-c \cdot e_n, \frac{-r^2}{c \cdot e_n}\right\}.$$

Let $y = c + te_n$. Then $y \cdot e_n = c \cdot e_n + t > 0$ because $t > -c \cdot e_n$. Hence $y \in E^n_+$.

$$J_S(y) = \hat{I}_{cr}(y) = \frac{r^2}{\|te_n\|^2}(te_n) + c = \frac{r^2}{t}e_n + c.$$

Thus, $J_S(y) \cdot e_n = \frac{r^2}{t} + c \cdot e_n < 0$ because $t > \frac{-r^2}{c \cdot e_n}$. Thus $J_S(y) \notin E^n_+$. Hence, $J_S(E^n_+) \neq E^n_+$. So $J_S \notin \mathrm{Mob}(E^n_+)$.

$\square$

We now introduce a useful device called the cross-ratio. We prove a key lemma about the cross-ratio that reveals the "rigidity" of Mobius transformations.

Def Let $(\hat{E}^n)^4 = \hat{E}^n \times \hat{E}^n \times \hat{E}^n \times \hat{E}^n$ and let $F = \{(u\,v\,x\,y) \in (\hat{E}^n)^4 : u = v \text{ or } x = y\}$. The cross-ratio is the function

$$(u,v,x,y) \longmapsto [u,v,x,y] : (\hat{E}^n)^4 - F \to [0\ \infty)$$

which is determined by the following equations. Let $(u,v,x,y) \in (\hat{E}^n)^4 \setminus F$.

$$[u,v,x,y] = \begin{cases} \dfrac{\|u-x\|\,\|v-y\|}{\|u-v\|\,\|x-y\|} & \text{if } u,v,x,y \in E^n \\[2ex] \dfrac{\|v-y\|}{\|x-y\|} & \text{if } u=\infty \text{ and } v,x,y \in E^n \\[2ex] \dfrac{\|u-x\|}{\|x-y\|} & \text{if } v=\infty \text{ and } u,x,y \in E^n \\[2ex] \dfrac{\|v-y\|}{\|u-v\|} & \text{if } x=\infty \text{ and } u,v,y \in E^n \\[2ex] \dfrac{\|u-x\|}{\|u-v\|} & \text{if } y=\infty \text{ and } u,v,x \in E^n \\[2ex] 0 & \text{if } u=x=\infty \text{ and } v,y \in E^n \\ & \text{or } v=y=\infty \text{ and } u,x \in E^n \\[2ex] 1 & \text{if } u=y=\infty \text{ and } v,x \in E^n \\ & \text{or } v=x=\infty \text{ and } u,y \in E^n \end{cases}$$

Lemma 4.10. The cross-ration

$$(u,v,x,y) \mapsto [u,v,x,y] : (\hat{\mathbb{E}}^n)^4 - F \to [0,\infty)$$

is a continuous function.

Proof We need only consider the case in which one or two of u,v,x,y equals ∞.

For $u,v,x,y \in \mathbb{E}^n$,

$$[u,v,x,y] \geq \frac{(\|u\| - \|x\|)\,\|v-y\|}{\|u\| + \|v\|\,\|x-y\|} = \frac{\left(1-\frac{\|x\|}{\|u\|}\right)\|v-y\|}{\left(1+\frac{\|v\|}{\|u\|}\right)\|x-y\|}$$

Thus, $[u,v,x,y] \to \frac{\|v-y\|}{\|x-y\|}$ as $\|u\| \to \infty$.

Hence cross-ratio is continuous at (∞, v, x, y) when $v,x,y \in \mathbb{E}^n$. Similar arguments prove continuity at points of $\mathbb{E}^n \times \{\infty\} \times \mathbb{E}^n \times \mathbb{E}^n$, $\mathbb{E}^n \times \mathbb{E}^n \times \{\infty\} \times \mathbb{E}^n$ and $\mathbb{E}^n \times \mathbb{E}^n \times \mathbb{E}^n \times \{\infty\}$.

For $u,v,x,y \in \mathbb{E}^n$,

$$[u,v,x,y] = \frac{\left\|\frac{u}{\|u\|\|x\|} - \frac{x}{\|u\|\|x\|}\right\|\,\|v-y\|}{\left\|\frac{u}{\|u\|} - \frac{v}{\|u\|}\right\|\,\left\|\frac{x}{\|x\|} - \frac{y}{\|x\|}\right\|}.$$ Thus,

$[u,v,x,y] \to 0$ as $\|u\| \to \infty$ and $\|x\| \to \infty$. Hence, cross-ratio is continuous at points of $\{\infty\} \times \mathbb{E}^n \times \{\infty\} \times \mathbb{E}^n$. A similar argument proves continuity at points of

$\mathbb{E}^n \times \{\infty\} \times \mathbb{E}^n \times \{\infty\}$.

For $u, v, x, y \in \mathbb{E}^n$,

$$[u, v, x, y] = \frac{\left\| \frac{u}{\|u\|} - \frac{x}{\|u\|} \right\| \left\| \frac{v}{\|y\|} - \frac{y}{\|y\|} \right\|}{\left\| \frac{u}{\|u\|} - \frac{v}{\|u\|} \right\| \left\| \frac{x}{\|y\|} - \frac{y}{\|y\|} \right\|}.$$ Thus

$[u, v, x, y] \to 1$ as $\|u\| \to \infty$ and $\|y\| \to \infty$. Hence, cross-ratio is continuous at points of $\{\infty\} \times \mathbb{E}^n \times \mathbb{E}^n \times \{\infty\}$. A similar argument proves continuity at points of $\mathbb{E}^n \times \{\infty\} \times \{\infty\} \times \mathbb{E}^n$. $\square$

Next we prove Mobius transformations preserve cross-ratio.

<u>Theorem 4.11.</u> Mobius transformations preserve cross-ratio. In other words, if $\phi \in \mathrm{Mob}(\hat{\mathbb{E}}^n)$ and $(u, v, x, y) \in (\hat{\mathbb{E}}^n)^4 - F$, then

$$[\phi(u), \phi(v), \phi(x), \phi(y)] = [u, v, x, y].$$

<u>Proof</u> Clearly extended reflections preserve cross ratios because they are isometries on $\mathbb{E}^n$ and send ∞ to ∞.

To show that extended inversions preserve cross-ratio, we establish:

<u>Lemma 4.12</u> For $c \in \mathbb{E}^n$, $r>0$ and $x, y \in \mathbb{E}^n - \{c\}$,

$$\| I_{cr}(x) - I_{cr}(y) \| = \frac{r^2 \|x-y\|}{\|x-c\| \, \|y-c\|}$$

<u>Proof</u> $\| I_{cr}(x) - I_{cr}(y) \| =$

$$\left\| \frac{r^2}{\|x-c\|^2}(x-c) - \frac{r^2}{\|y-c\|^2}(y-c) \right\| =$$

$$\frac{r^2}{\|x-c\|\,\|y-c\|} \left\| \frac{\|y-c\|}{\|x-c\|}(x-c) - \frac{\|x-c\|}{\|y-c\|}(y-c) \right\| =$$

$$\frac{r^2}{\|x-c\|\,\|y-c\|} \sqrt{\|y-c\|^2 - 2(x-c)\cdot(y-c) + \|x-c\|^2} =$$

$$\frac{r^2}{\|x-c\|\,\|y-c\|} \|(y-c)-(x-c)\| = \frac{r^2\|x-y\|}{\|x-c\|\,\|y-c\|}. \quad \square$$

Returning to the proof of Theorem 4.11: suppose $(u,v,x,y) \in (\mathbb{E}^n - \{c\})^4 - F$. Then by Lemma 4.12,

$$[\hat{I}_{cr}(u), \hat{I}_{cr}(v), \hat{I}_{cr}(x), \hat{I}_{cr}(y)] =$$

$$\frac{\left(\frac{r^2\|u-x\|}{\|u-c\|\,\|x-c\|}\right)\left(\frac{r^2\|v-y\|}{\|v-c\|\,\|y-c\|}\right)}{\left(\frac{r^2\|u-v\|}{\|u-c\|\,\|v-c\|}\right)\left(\frac{r^2\|x-y\|}{\|x-c\|\,\|y-c\|}\right)} = [u,v,x,y].$$

For $v, x, y \in \mathbb{E}^n - \{c\}$,

$[\hat{I}_{cr}(c), \hat{I}_{cr}(v), \hat{I}_{cr}(x), \hat{I}_{cr}(y)] = [\infty, \hat{I}_{cr}(v), \hat{I}_{cr}(x), \hat{I}_{cr}(y)] =$

$$\frac{\|\hat{I}_{cr}(v) - \hat{I}_{cr}(y)\|}{\|\hat{I}_{cr}(x) - \hat{I}_{cr}(y)\|} = \frac{\left(\frac{r^2 \|v-y\|}{\|v-c\| \, \|y-c\|}\right)}{\left(\frac{r^2 \|x-y\|}{\|x-c\| \, \|y-c\|}\right)} =$$

$$\frac{\|c-x\| \, \|v-y\|}{\|c-v\| \, \|x-y\|} = [c, v, x, y].$$ Hence,

$[\hat{I}_{cr}(\infty), \hat{I}_{cr}(v), \hat{I}_{cr}(x), \hat{I}_{cr}(y)] = [c, \hat{I}_{cr}(v), \hat{I}_{cr}(x), \hat{I}_{cr}(y)] =$

$[\hat{I}_{cr}(c), \hat{I}_{cr} \circ \hat{I}_{cr}(v), \hat{I}_{cr} \circ \hat{I}_{cr}(x), \hat{I}_{cr} \circ \hat{I}_{cr}(y)] = [\infty, v, x, y]$

This proves $[\hat{I}_{cr}(u), \hat{I}_{cr}(v), \hat{I}_{cr}(x), \hat{I}_{cr}(y)] = [u\,v\,x\,y]$

for $(u, v, x, y) \in \{c, \infty\} \times (\mathbb{E}^n - \{c\}) \times (\mathbb{E}^n - \{c\}) \times (\mathbb{E}^n - \{c\}) - F$.

Similar arguments prove

$[\hat{I}_{cr}(u), \hat{I}_{cr}(v), \hat{I}_{cr}(x), \hat{I}_{cr}(y)] = [u, v, x, y]$ for

(u, v, x, y) belonging to any of the sets

$(\mathbb{E}^n - \{c\}) \times \{c\,\infty\} \times (\mathbb{E}^n - \{c\})^2 - F$,

$(\mathbb{E}^n - \{c\})^2 \times \{c, \infty\} \times (\mathbb{E}^n - \{c\}) - F$ and

$(\mathbb{E}^n - \{c\})^3 \times \{c\,\infty\} - F$.

For $v, y \in \mathbb{E}^n - \{c\}$,

$[\hat{I}_{cr}(\infty), \hat{I}_{cr}(v), \hat{I}_{cr}(\infty), \hat{I}_{cr}(y)] = [c, \hat{I}_{cr}(v), c, \hat{I}_{cr}(y)] =$

$0 = [\infty, v, \infty, y]$. ~~Also~~ Also $[\hat{I}_{cr}(c), \hat{I}_{cr}(v), \hat{I}_{cr}(c), \hat{I}_{cr}(y)] =$

-4.48-

$[\infty, \hat{I}_{cr}(v), \infty, \hat{I}_{cr}(y)] = 0 = [c, v, c, y]$.

Furthermore, $[\hat{I}_{cr}(\infty), \hat{I}_{cr}(v), \hat{I}_{cr}(c), \hat{I}_{cr}(y)] =$

$$[c, \hat{I}_{cr}(v), \infty, \hat{I}_{cr}(y)] = \frac{\|\hat{I}_{cr}(v) - \hat{I}_{cr}(y)\|}{\|c - \hat{I}_{cr}(v)\|} =$$

$$\frac{\left(\dfrac{r^2\|v-y\|}{\|v-c\|\,\|y-c\|}\right)}{\left\|\dfrac{r^2}{\|v-c\|^2}(v-c)\right\|} = \frac{\|v-y\|}{\|c-y\|} = [\infty, v, c, y].$$

Hence, $[\hat{I}_{cr}(c), \hat{I}_{cr}(v), \hat{I}_{cr}(\infty), \hat{I}_{cr}(y)] =$

$[\infty, \hat{I}_{cr}(v), c, \hat{I}_{cr}(y)] = [\hat{I}_{cr}(\infty), \hat{I}_{cr} \circ \hat{I}_{cr}(v), \hat{I}_{cr}(c), \hat{I}_{cr} \circ \hat{I}_{cr}(y)] =$

$[c, v, \infty, y]$. This proves $[\hat{I}_{cr}(u), \hat{I}_{cr}(v), \hat{I}_{cr}(x), \hat{I}_{cr}(y)] =$

$[u, v, x, y]$ for $(u v x y) \in \{c, \infty\} \times (\mathbb{E}^n - \{c\}) \times \{c, \infty\} \times (\mathbb{E}^n - \{c\})$

A similar argument proves $[\hat{I}_{cr}(u), \hat{I}_{cr}(v), \hat{I}_{cr}(x), \hat{I}_{cr}(y)] =$

$[u, v, x y]$ for $(u v x y) \in (\mathbb{E}^n - \{c\}) \times \{c, \infty\} \times (\mathbb{E}^n - \{c\}) \times \{c, \infty\}$.

For $v, x \in \mathbb{E}^n - \{c\}$,

$$[\hat{I}_{cr}(\infty), \hat{I}_{cr}(v), \hat{I}_{cr}(x), \hat{I}_{cr}(\infty)] = [c, \hat{I}_{cr}(v), \hat{I}_{cr}(x), c] =$$

$$\frac{\|c - \hat{I}_{cr}(x)\|\,\|\hat{I}_{cr}(v) - c\|}{\|c - \hat{I}_{cr}(v)\|\,\|\hat{I}_{cr}(x) - c\|} = 1 = [\infty, v, x, \infty].$$ Also

$$[\hat{I}_{cr}(c), \hat{I}_{cr}(v), \hat{I}_{cr}(x), \hat{I}_{cr}(c)] = [\infty, \hat{I}_{cr}(v), \hat{I}_{cr}(x), \infty] =$$

$$1 = \frac{\|c-x\|\,\|v-c\|}{\|c-v\|\,\|x-c\|} = [c, v, x, c].$$ Furthermore,

$$[\hat{I}_{cr}(c), \hat{I}_{cr}(v), \hat{I}_{cr}(x), \hat{I}_{cr}(\infty)] = [\infty, \hat{I}_{cr}(v), \hat{I}_{cr}(x), c] =$$

$$\frac{\|\hat{I}_{cr}(v) - c\|}{\|\hat{I}_{cr}(x) - c\|} = \frac{\left\|\frac{r^2}{\|v-c\|^2}(v-c)\right\|}{\left\|\frac{r^2}{\|x-c\|^2}(x-c)\right\|} = \frac{\|x-c\|}{\|v-c\|} = [c, v, x, \infty]$$

Hence, $[\hat{I}_{cr}(\infty), \hat{I}_{cr}(v), \hat{I}_{cr}(x), \hat{I}_{cr}(c)] = [c, \hat{I}_{cr}(v), \hat{I}_{cr}(x), \infty] =$

$[\hat{I}_{cr}(c), \hat{I}_{cr} \circ \hat{I}_{cr}(v)\ \hat{I}_{cr} \circ \hat{I}_{cr}(x), \hat{I}_{cr}(\infty)] = [\infty, v, x, c]$

This proves $[\hat{I}_{cr}(u), \hat{I}_{cr}(v), \hat{I}_{cr}(x), \hat{I}_{cr}(y)] = [u, v, x, y]$
for $(u, v, x, y) \in \{c, \infty\} \times (\mathbb{E}^n - \{c\}) \times (\mathbb{E}^n - \{c\}) \times \{c, \infty\}$.
A similar argument shows $[\hat{I}_{cr}(u), \hat{I}_{cr}(v), \hat{I}_{cr}(x), \hat{I}_{cr}(y)]$
$= [u, v, x, y]$ for all $(u v x y) \in (\mathbb{E}^n - \{c\}) \times \{c, \infty\} \times \{c, \infty\} \times (\mathbb{E}^n - \{c\})$.

This completes the proof that
$[\hat{I}_{cr}(u), \hat{I}_{cr}(v), \hat{I}_{cr}(x), \hat{I}_{cr}(y)] = [u, v, x, y]$
for all $(u, v, x, y) \in (\hat{\mathbb{E}}^n)^4 - F$.

Thus, all extended reflections and all extended inversions preserve cross-ratio. Clearly compositions of functions that preserve cross-ratios also preserve cross-ratios. Hence, all Mobius transformations preserve cross-ratios. □

Next we prove a key lemma concerning cross-ratio preserving functions. We begin with a definition.

<u>Definition</u>. A function $f: \mathbb{E}^m \to \mathbb{E}^n$ is a <u>similarity</u> if there is a $k > 0$ such that $\|f(x) - f(y)\| = k\|x - y\|$ for all $x, y \in \mathbb{E}^m$.

<u>Key Lemma 4.13</u> If $\phi: \hat{\mathbb{E}}^n \to \hat{\mathbb{E}}^n$ is an injective cross-ratio preserving function and $\phi(\infty) = \infty$, then $\phi | \mathbb{E}^n$ is a similarity.

<u>Proof</u> Let $u, v, x, y \in \mathbb{E}^n$ such that $u \neq v$, $u \neq x$ and $x \neq y$. Since

$$[u, \infty, x, y] = [\phi(u), \phi(\infty), \phi(x), \phi(y)] = [\phi(u), \infty, \phi(x), \phi(y)],$$

then

$$\frac{\|u - x\|}{\|x - y\|} = \frac{\|\phi(u) - \phi(x)\|}{\|\phi(x) - \phi(y)\|}.$$

Hence

$$\frac{\|\phi(x) - \phi(y)\|}{\|x - y\|} = \frac{\|\phi(u) - \phi(x)\|}{\|u - x\|}.$$

Since

$$[u, v, x, \infty] = [\phi(u), \phi(v), \phi(x), \phi(\infty)] = [\phi(u), \phi(v), \phi(x), \infty],$$

then

$$\frac{\|u - x\|}{\|u - v\|} = \frac{\|\phi(u) - \phi(x)\|}{\|\phi(u) - \phi(v)\|}.$$

Hence

$$\frac{\|\phi(u) - \phi(v)\|}{\|u - v\|} = \frac{\|\phi(u) - \phi(x)\|}{\|u - x\|}.$$

Therefore, $\dfrac{\|\phi(x)-\phi(y)\|}{\|x-y\|} = \dfrac{\|\phi(u)-\phi(v)\|}{\|u-v\|}$.

It follows that there is a constant $k>0$ such that $\dfrac{\|\phi(x)-\phi(y)\|}{\|x-y\|} = k$ for all $x, y \in \mathbb{E}^n$ such that $x \neq y$. Hence, $\phi | \mathbb{E}^n$ is a similarity. $\square$

We now derive consequences of the Key Lemma.

Theorem 4.14. An injective function $\phi : \hat{\mathbb{E}}^n \to \hat{\mathbb{E}}^n$ is a Mobius transformation if and only ϕ preserves cross-ratios.

Proof Every Mobius transformation preserves cross-ratios by Theorem 4.11.

Suppose $\phi : \hat{\mathbb{E}}^n \to \hat{\mathbb{E}}^n$ is injective and preserves cross-ratios. We will prove below that if $\phi(\infty) = \infty$, then $\phi \in Mob(\hat{\mathbb{E}}^n)$. Assume we have established this assertion. Suppose $\phi(\infty) = c \neq \infty$. Then $\hat{I}_{c,1} \circ \phi(\infty) = \infty$. Also $\hat{I}_{c,1}$ preserves cross-ratio by Theorem 4.11. Hence, $\hat{I}_{c,1} \circ \phi$ preserves cross-ratio. It will

follow from our assertion that $\hat{I}_{c1} \circ \phi \in Mob(\hat{\mathbb{E}}^n)$.
But then $\phi = \hat{I}_{c1} \circ (\hat{I}_{c1} \circ \phi) \in Mob(\hat{\mathbb{E}}^n)$.
Hence, we can assume $\phi(\infty) = \infty$.

Since $\phi(\infty) = \infty$, then the Key Lemma implies $\phi | \mathbb{E}^n$ is a similarity. Thus there is a $k > 0$ such that $\|\phi(x) - \phi(y)\| = k\|x-y\|$ for all $x, y \in \mathbb{E}^n$. Define $\psi : \hat{\mathbb{E}}^n \to \hat{\mathbb{E}}^n$ by

$$\psi(x) = \begin{cases} \frac{1}{k}\phi(x) & \text{if } x \in \mathbb{E}^n \\ \infty & \text{if } x = \infty \end{cases}.$$

Then $\|\psi(x) - \psi(y)\| = \|\frac{1}{k}\phi(x) - \frac{1}{k}\phi(y)\| = \frac{1}{k}\|\phi(x) - \phi(y)\|$ $= \|x-y\|$ for all $x, y \in \mathbb{E}^n$. Thus $\psi | \mathbb{E}^n$ is an isometry. So there are reflections $Z_1, \dots, Z_k$ of $\mathbb{E}^n$ such that $\psi = \hat{Z}_1 \circ \dots \circ \hat{Z}_k$.

Thus, $\phi = k\psi = \hat{D}_{0k} \circ \psi = \hat{I}_{0\sqrt{k}} \circ \hat{I}_{01} \circ \hat{Z}_1 \circ \dots \circ \hat{Z}_k$.

Therefore $\phi \in Mob(\hat{\mathbb{E}}^n)$. $\square$

A Characterization Theorem 4,15.

Let $\phi \in \mathrm{Mob}(\hat{\mathbb{E}}^n)$.

a) If $\phi(\infty) \neq \infty$, then $\phi = \hat{\tau} \circ \hat{D}_{ok}$ where $k > 0$ and $\tau : \mathbb{E}^n \to \mathbb{E}^n$ is an isometry;

b) If $\phi(\infty) \neq \infty$, then $\phi = \hat{\tau} \circ \hat{I}_{cr}$ where $\tau : \mathbb{E}^n \to \mathbb{E}^n$ is an isometry, $\phi(c) = \infty$ and $r > 0$ such that $S(cr)$ is the only sphere centered at c that isn't expanded or contracted by ϕ. ($S(cr)$ is called the <u>isometric sphere</u> of ϕ.)

<u>Proof of a)</u> Assume $\phi(\infty) = \infty$. Then the Key Lemma implies $\phi | \mathbb{E}^n$ is a similarity. Hence there is a $k > 0$ such that $\sigma = \frac{1}{k}\phi | \mathbb{E}^n$ is an isometry of $\mathbb{E}^n$. So $\phi | \mathbb{E}^n = k\sigma = D_{ok} \circ \sigma$. Now $\sigma = T_a \circ \rho$ where $\rho : \mathbb{E}^n \to \mathbb{E}^n$ is an orthogonal map and $T_a(x) = x + a$. In this case, $a = \sigma(0)$. Thus, $\phi | \mathbb{E}^n = D_{ok} \circ T_a \circ \rho$. Observe that $D_{ok} \circ T_a \circ \rho(x) = k(\rho(x) + a) = \rho(kx) + ka = T_{ka} \circ \rho \circ D_{ok}(x)$. Define the isometry $\tau : \mathbb{E}^n \to \mathbb{E}^n$ by $\tau = T_{ka} \circ \rho$. Then $\phi | \mathbb{E}^n = \tau \circ D_{ok}$. Hence, $\phi = \hat{\tau} \circ \hat{D}_{ok}$. $\square$

Proof of b). Assume $\phi(\infty) \neq \infty$.
Let $c = \phi^{-1}(\infty) \in \mathbb{E}^n$. Then $\phi \circ \hat{I}_{c,1} \in Mob(\hat{\mathbb{E}}^n)$ and $\phi \circ \hat{I}_{c1}(\infty) = \infty$. Therefore, the Key Lemma implies $\phi \circ \hat{I}_{c,1} | \mathbb{E}^n$ is a similarity. Hence, there is a $k > 0$ such that $\sigma = (1/k) \phi \circ \hat{I}_{c,1} | \mathbb{E}^n$ is an isometry of $\mathbb{E}^n$. Therefore, $\phi \circ \hat{I}_{c1} | \mathbb{E}^n = D_{0,k} \circ \sigma$. As above, $\sigma = T_a \circ \rho$ where $\rho : \mathbb{E}^n \to \mathbb{E}^n$ is an isometry, $T_a(x) = x + a$ and $a = \sigma(0)$. Thus,

$$\phi \circ \hat{I}_{c,1} | \mathbb{E}^n = D_{0k} \circ T_a \circ \rho = T_{ka} \circ \rho \circ D_{0k}.$$

Hence,

$$\phi | \mathbb{E}^n - \{c\} = \phi \circ I_{c1} \circ I_{c1} = T_{ka} \circ \rho \circ D_{0k} \circ I_{c1}$$

Let $r = \sqrt{k}$. Then for $x \in \mathbb{E}^n - \{c\}$,

$$D_{0k} \circ I_{c1}(x) = k\left(\frac{1}{\|x-c\|^2}(x-c) + c\right) =$$

$$\left(\frac{r^2}{\|x-c\|^2}(x-c) + c\right) + (r^2-1)c = T_{(r^2-1)c} \circ I_{c,r}(x).$$

Thus, $\phi | \mathbb{E}^n - \{c\} = T_{r^2 a} \circ \rho \circ T_{(r^2-1)c} \circ I_{cr}$.
Define the isometry $\tau : \mathbb{E}^n \to \mathbb{E}^n$ by $\tau = T_{r^2 a} \circ \rho \circ T_{(r^2-1)c}$.
Then $\phi | \mathbb{E}^n - \{c\} = \tau \circ I_{cr}$. Hence $\phi = \hat{\tau} \circ \hat{I}_{c,r}$. $\square$

The following theorem is a consequence of the Key Lemma which reveals the rigidity of Mobius Transformations.

<u>Theorem 4.16.</u> If $\phi \in Mob(\hat{\mathbb{E}}^n)$ and $\phi | \mathbb{E}^{n-1} \times \{0\} = id$, then either $\phi = id$ or $\phi = \hat{Z}_{e_n,0}$.

<u>Proof</u> Assume $\phi \in Mob(\hat{\mathbb{E}}^n)$, $\phi | \mathbb{E}^{n-1} \times \{0\} = id$ and $\phi \neq id$. Since ∞ is a limit point of $\mathbb{E}^{n-1} \times \{0\}$ and ϕ is continuous, then $\phi(\infty) = \infty$. Hence, $\phi | \mathbb{E}^n$ is a similarity by the Key Lemma. So there is a $k > 0$ such that $\|\phi(x) - \phi(y)\| = k\|x-y\|$ for all $x, y \in \mathbb{E}^n$. Since $\phi | \mathbb{E}^{n-1} \times \{0\} = id$, then $k = 1$. Thus $\phi | \mathbb{E}^n$ is an isometry. Since $\phi(0) = 0$, then $\phi | \mathbb{E}^n$ is an orthogonal map. Since $\phi(e_i) = e_i$ for $1 \leq i \leq n-1$ and $\phi(e_1), \ldots, \phi(e_n)$ is an orthonormal basis, then $\phi(e_n) = \pm e_n$. If $\phi(e_n) = e_n$, then $\phi = id$. Therefore, $\phi(e_n) = -e_n$. Then $\phi(e_i) = Z_{e_n,0}(e_i)$ for $1 \leq i \leq n$. Hence, $\phi = \hat{Z}_{e_n,0}$. $\square$

The isometry group of hyperbolic space $\mathcal{I}(\mathbb{H}^n)$ enjoys a rigidity property like the property of $\mathrm{Mob}(\hat{\mathbb{E}}^n)$ revealed by Theorem 4.46. We now formulate a theorem stating this property of $\mathcal{I}(\mathbb{H}^n)$. This theorem should have been included in Chapter 2 right after Theorem 2.3

<u>Theorem 2.3½</u> If V is an n-dimensional vector subspace of $\mathbb{M}^{n+1}$ such that that $V \cap \mathbb{H}^n \neq \emptyset$ and $g \in \mathcal{I}(\mathbb{H}^n)$ such that $g \restriction V \cap \mathbb{H}^n = id$, then either $g = id$ or $g = Z_u | \mathbb{H}^n$ where u is a spacelike unit vector in $\mathbb{M}^{n+1}$ such that $V = \{x \in \mathbb{M}^{n+1} : x \circ u = 0\}$.

<u>Proof</u> Choose $u_{n+1} \in V \cap \mathbb{H}^n$. Enlarge u_{n+1} to an orthonormal basis $u_1, \ldots, u_{n+1}$ for $\mathbb{M}^{n+1}$ such that $u_2, \ldots, u_{n+1} \in V$. Then $V = \{x \in \mathbb{M}^{n+1} : x \circ u_1 = 0\}$. For $2 \le i \le n$, replace u_i by $-u_i$ if necessary to make $u_i \circ e_{n+1} \leq 0$. Then $u_i + \sqrt{2}\, u_{n+1} \in V \cap \mathbb{H}^n$ for $2 \le i \le n$ because $(u_i + \sqrt{2}\, u_{n+1}) \circ (u_i + \sqrt{2}\, u_{n+1}) = -1$ and $(u_i + \sqrt{2}\, u_{n+1}) \circ e_{n+1} < 0$. Then $u_2 + \sqrt{2}\, u_{n+1}, \ldots, u_n + \sqrt{2}\, u_{n+1}, u_{n+1}$ is a basis for V.

Theorem 2.3 implies there is a $\bar{g} \in O^+(\mathbb{M}^{n+1})$

such that $\bar{g}|\mathbb{H}^n = g$. Thus,

$$\bar{g}(u_i + \sqrt{2}\,u_{n+1}) = g(u_i + \sqrt{2}\,u_{n+1}) = u_i + \sqrt{2}\,u_{n+1}$$

for $2 \le i \le n$ and

$$\bar{g}(u_{n+1}) = g(u_{n+1}) = u_{n+1}.$$

Since $u_2 + \sqrt{2}\,u_{n+1}, \ldots, u_n + \sqrt{2}\,u_{n+1}, u_{n+1}$ is a basis for V and $\bar{g}$ is linear, then $\bar{g}|W = id$. Thus $\bar{g}(u_i) = u_i$ for $2 \le i \le n$. Since $\bar{g}(u_1), \ldots, \bar{g}(u_{n+1})$ is an orthonormal basis for $\mathbb{M}^{n+1}$, it follows that $\bar{g}(u_1) = \pm u_1$. If $\bar{g}(u_1) = u_1$, then $\bar{g} = id_{\mathbb{M}^{n+1}}$ and $g = \bar{g}|\mathbb{H}^n = id_{\mathbb{H}^n}$. If $\bar{g}(u_1) = -u_1$, then $\bar{g}(u_i) = Z_{u_1}(u_i)$ for $1 \le i \le n+1$. Then $\bar{g} = Z_{u_1}$ and $g = \bar{g}|\mathbb{H}^n = Z_{u_1}|\mathbb{H}^n$. $\square$

Theorem 4.16 yields the following useful results.

<u>Corollary 4.17</u>. If $S, T \in \Sigma^n$ and $\phi \in Mob(\hat{\mathbb{E}}^n)$ such that $\phi(S) = T$, then $J_T = \phi \circ J_S \circ \phi^{-1}$.

Proof Let $P = \hat{P}(e_n, 0) = (\mathbb{E}^{n-1} \times \{0\}) \cup \{\infty\}$. Then $P \in \Sigma^n$ and $J_P = Z_{e_n, 0}$.

First we prove that if $S \in \Sigma^n$ and $\phi \in Mob(\hat{\mathbb{E}}^n)$ such that $\phi(P) = S$, then $J_S = \phi \circ J_P \circ \phi^{-1}$. Observe that $\phi^{-1} \circ J_S \circ \phi \in Mob(\hat{\mathbb{E}}^n)$ and $\phi^{-1} \circ J_S \circ \phi | P = id$. Hence, Theorem 4.16 implies either $\phi^{-1} \circ J_S \circ \phi = id$ or $\phi^{-1} \circ J_S \circ \phi = J_P$. Since $J_S \neq id$, then $\phi^{-1} \circ J_S \circ \phi \neq id$. Therefore, $\phi^{-1} \circ J_S \circ \phi = J_P$. Consequently, $J_S = \phi \circ J_P \circ \phi^{-1}$.

Now consider the general case in which $S, T \in \Sigma^n$ and $\phi \in Mob(\hat{\mathbb{E}}^n)$ such that $\phi(S) = T$. Theorem 4.5 provides a $\psi \in Mob(\hat{\mathbb{E}}^n)$ such that $\psi(P) = S$. Then $\phi \circ \psi \in Mob(\hat{\mathbb{E}}^n)$ and $\phi \circ \psi(P) = T$. Now the result of the preceding paragraph implies $J_S = \psi \circ J_P \circ \psi^{-1}$ and $J_T = (\phi \circ \psi) \circ J_P \circ (\phi \circ \psi)^{-1}$. Thus, $J_P = \psi^{-1} \circ J_S \circ \psi$ and $J_T = \phi \circ \psi \circ J_P \circ \psi^{-1} \circ \phi^{-1} = \phi \circ \psi \circ (\psi^{-1} \circ J_S \circ \psi) \circ \psi^{-1} \circ \phi^{-1}$ $= \phi \circ J_S \circ \phi^{-1}$. $\square$

Corollary 4.18 Suppose $S, T \in \Sigma^n$ and $\phi \in Mob(\hat{\mathbb{E}}^n)$ such that $\phi(S) = T$. If x and $y \in \hat{\mathbb{E}}^n$ such that $J_S(x) = y$, then $J_T(\phi(x)) = \phi(y)$.

Proof. Corollary 4.17 implies $J_T = \phi \circ J_S \circ \phi^{-1}$. Hence, $J_T \circ \phi = \phi \circ J_S$. Therefore, $J_T(\phi(x)) = \phi(J_S(x)) = \phi(y)$. □

Corollary 4.19. If $S \in \Sigma^n$ and $\phi \in Mob(\hat{\mathbb{E}}^n)$ such that $\phi | S = id$, then either $\phi = id$ or $\phi = J_S$.

Proof Let $P = \hat{P}(e_n, 0) = (\mathbb{E}^{n-1} \times \{0\}) \cup \{\infty\}$. Then $P \in \Sigma^n$ and $J_P = \hat{Z}_{e_n, 0}$.

Theorem 4.5 provides a $\psi \in Mob(\hat{\mathbb{E}}^n)$ such that $\psi(P) = S$. Hence, $\psi^{-1} \circ \phi \circ \psi \in Mob(\hat{\mathbb{E}}^n)$ and $\psi^{-1} \circ \phi \circ \psi | P = id$. Now Theorem 4.16 implies that either $\psi^{-1} \circ \phi \circ \psi = id$ or $\psi^{-1} \circ \phi \circ \psi = J_P$. Observe that $\psi^{-1} \circ \phi \circ \psi = id$ implies $\phi = id$, and $\psi^{-1} \circ \phi \circ \psi = J_P$ implies $\phi = \psi \circ J_P \circ \psi^{-1}$ which implies $\phi = J_S$ by Corollary 4.17. We conclude that either $\phi = id$ or $\phi = J_S$. □

Corollary 4.20 Suppose $S \in \Sigma^n$, V is a component of $\hat{\mathbb{E}}^n - S$ and ϕ and $\psi \in \mathrm{Mob}(V)$. If $\phi|S = \psi|S$, then $\phi = \psi$.

Proof $\psi^{-1} \circ \phi \in \mathrm{Mob}(\hat{\mathbb{E}}^n)$ and $\psi^{-1} \circ \phi | S = id$. Hence, Corollary 4.19 implies that either $\psi^{-1} \circ \phi = id$ or $\psi^{-1} \circ \phi = J_S$. Since $\psi^{-1} \circ \phi(V) = V$ while $J_S(V) \neq V$, then then $\psi^{-1} \circ \phi \neq J_S$. Therefore, $\psi^{-1} \circ \phi = id$. Consequently, $\phi = \psi$. $\square$

Corollary 4.21. Suppose $S, T \in \Sigma^n$ and V is a component of $\hat{\mathbb{E}}^n - S$. Then $J_T \in \mathrm{Mob}(V)$ if and only if $S - \{\infty\}$ is orthogonal to $T - \{\infty\}$.

Proof Let $P = \hat{P}(e_n, 0) = (\mathbb{E}^{n-1} \times \{0\}) \cup \{\infty\}$. Then $J_P = \hat{Z}_{e_n, 0}$. Hence, $J_P(\mathbb{E}^n_+) = \mathbb{E}^n_-$. Theorem 4.5 provides a $\phi \in \mathrm{Mob}(\hat{\mathbb{E}}^n)$ such that $\phi(P) = S$. Either $\phi(\mathbb{E}^n_+) = V$ or $\phi(\mathbb{E}^n_-) = V$. In the latter case $\phi \circ J_P(\mathbb{E}^n_+) = V$. So if we replace ϕ by $\phi \circ J_P$ if necessary, we can assume $\phi(\mathbb{E}^n_+) = V$.

Since ϕ is a composition of extended reflections and extended inversions which are conformal, then ϕ is conformal (on $\hat{\mathbb{E}}^n - \phi^{-1}(\infty)$). Therefore, Lemma 4.7 implies ϕ preserves the orthogonality of elements of Σ^n. The same is true of ϕ^{-1}. Let $Q = \phi^{-1}(T)$. Then Theorem 4.9 implies $J_Q \in Mob(\mathbb{E}^n_+)$ if and only if $P - \{\infty\}$ is orthogonal to $Q - \{\infty\}$, and Corollary 4.17 implies $J_Q = \phi^{-1} \circ J_T \circ \phi$. Hence, the following statements are equivalent

- $S - \{\infty\}$ is orthogonal to $T - \{\infty\}$.
- $P - \{\infty\}$ is orthogonal to $Q - \{\infty\}$.
- $J_Q \in Mob(\mathbb{E}^n_+)$.
- $J_Q(\mathbb{E}^n_+) = \mathbb{E}^n_+$.
- $(\phi^{-1} \circ J_T \circ \phi)(\phi^{-1}(V)) = \phi^{-1}(V)$.
- $\phi^{-1} \circ J_T(V) = \phi^{-1}(V)$.
- $J_T(V) = V$
- $J_T \in Mob(V)$. $\square$

<u>Definition</u> For $x \in \hat{\mathbb{E}}^n$, define $\bar{x} \in \hat{\mathbb{E}}^{n+1}$ by

$$\bar{x} = \begin{cases} (x_1, \ldots, x_n, 0) & \text{if } x = (x_1, \ldots, x_n) \\ \infty & \text{if } x = \infty . \end{cases}$$

For $S \in \Sigma^n$, define $\bar{S} \in \Sigma^{n+1}$ by

$$\bar{S} = \begin{cases} \hat{P}(\bar{u}, a) & \text{if } S = \hat{P}(u, a) \\ S(\bar{c}, r) & \text{if } S = S(c, r) \end{cases}$$

For $S \in \Sigma^n$ let $\bar{J}_S$ denote the extended reflection/inversion in $\bar{S}$. In other words,

$$\bar{J}_S = J_{\bar{S}} = \begin{cases} \hat{Z}_{\bar{u}, a} & \text{if } S = P(u a) \\ \hat{I}_{\bar{c}, r} & \text{if } S = S(c r) \end{cases}.$$

If $\phi \in \mathrm{Mob}(\hat{\mathbb{E}}^n)$, the <u>Poincaré extension</u> $\bar{\phi}$ of ϕ is the element of $\mathrm{Mob}(\hat{\mathbb{E}}^{n+1})$ which is defined as follows. If $\phi = J_1 \circ \cdots \circ J_k$ where $J_1, \ldots, J_k$ are extended reflections/inversions of $\hat{\mathbb{E}}^n$, then $\bar{\phi} = \bar{J}_1 \circ \cdots \circ \bar{J}_k$.

Observe that for $x \in \hat{\mathbb{E}}^n$, $\hat{Z}_{\bar{u}, a}(\bar{x}) = \overline{\hat{Z}_{u,a}(x)}$ and $\hat{I}_{\bar{c}, r}(\bar{x}) = \overline{\hat{I}_{c,r}(x)}$. Thus, for $S \in \Sigma^n$ and $x \in \hat{\mathbb{E}}^n$, $\bar{J}_S(\bar{x}) = \overline{J_S(x)}$. Hence, for $\phi \in \mathrm{Mob}(\hat{\mathbb{E}}^n)$ and $x \in \hat{\mathbb{E}}^n$, $\bar{\phi}(\bar{x}) = \overline{\phi(x)}$.

Lemma 4.22. The Poincaré extension of an element of $\mathrm{Mob}(\hat{\mathbb{E}}^n)$ is well defined. In other words, if $\phi \in \mathrm{Mob}(\hat{\mathbb{E}}^n)$ and $\phi = J_1 \circ \cdots \circ J_k = J_1' \circ \cdots \circ J_\ell'$ where $J_1, \dots, J_k, J_1', \dots, J_\ell'$ are extended reflections/inversions of $\hat{\mathbb{E}}^n$, then $\overline{J_1} \circ \cdots \circ \overline{J_k} = \overline{J_1'} \circ \cdots \circ \overline{J_\ell'}$.

Proof. Let $S_1, \dots, S_k, S_1', \dots, S_\ell' \in \Sigma^n$ such that J_i is extended reflection/inversion in S_i for $1 \le i \le k$ and J_i' is extended reflection/inversion in S_i' for $1 \le i \le \ell$. With the aid of Theorem 4.8, observe that $\overline{S_1}, \dots, \overline{S_k}, \overline{S_1'}, \dots, \overline{S_\ell'}$ are orthogonal to $\mathbb{E}^n \times \{0\}$. Indeed, if S_i or S_i' is $\hat{P}(u,a)$ where $u \in \mathbb{E}^n$, then $\overline{S_i}$ or $\overline{S_i'}$ is $\hat{P}(\bar{u},a)$ where $\bar{u} \cdot e_{n+1} = 0$. Similarly, if S_i or S_i' is $S(c,r)$ where $c \in \mathbb{E}^n$, then $\overline{S_i}$ or $\overline{S_i'}$ is $S(\bar{c},r)$ where $\bar{c} \in \mathbb{E}^n \times \{0\}$. Consequently, Theorem 4.9 implies $\overline{J_1}, \dots, \overline{J_k}, \overline{J_1'}, \dots, \overline{J_\ell'} \in \mathrm{Mob}(\mathbb{E}^{n+1}_+)$. For $x \in$ [illegible]^n,

$$\overline{J_1} \circ \cdots \circ \overline{J_k}(\bar{x}) = \overline{J_1 \circ \cdots \circ J_k(x)} = \overline{\phi(x)} = \overline{J_1' \circ \cdots \circ J_\ell'(x)} =$$

$\overline{J_1'} \circ \cdots \circ \overline{J_k'}(\bar{x})$. Thus, $\overline{J_1} \circ \cdots \circ \overline{J_k} \,|\, \mathbb{E}^n \times \{0\} = \overline{J_1'} \circ \cdots \circ \overline{J_\ell'} \,|\, \mathbb{E}^n \times \{0\}$.

Hence, Corollary 4.21 implies $\overline{J_1} \circ \cdots \circ \overline{J_k} = \overline{J_1'} \circ \cdots \circ \overline{J_\ell'}$. $\square$

<u>Theorem 4.23.</u> If $\phi \in \text{Mob}(\hat{\mathbb{E}}^n)$, then $\bar{\phi} \in \text{Mob}(\mathbb{E}^{n+1}_+)$ and

$$\phi \mapsto \bar{\phi} : \text{Mob}(\hat{\mathbb{E}}^n) \to \text{Mob}(\mathbb{E}^{n+1}_+)$$

is an isomorphism.

<u>Proof</u> If $\phi \in \text{Mob}(\hat{\mathbb{E}}^n)$, then $\phi = J_1 \circ \cdots \circ J_k$ where J_i is the extended reflection/inversion in $S_i \in \Sigma^n$ for $1 \le i \le k$. As we observed in the proof of Lemma 4.22, each $\bar{S}_i$ is orthogonal to $\mathbb{E}^n \times \{0\}$ (by Theorem 4.8) and each $\bar{J}_i \in \text{Mob}(\mathbb{E}^{n+1}_+)$ (by Theorem 4.9). Thus, $\bar{\phi} = \bar{J}_1 \circ \cdots \circ \bar{J}_k \in \text{Mob}(\mathbb{E}^{n+1}_+)$.

The relation $\bar{\phi}(\bar{x}) = \overline{\phi(x)}$ for $x \in \mathbb{E}^n$ implies $\phi \mapsto \bar{\phi} : \text{Mob}(\hat{\mathbb{E}}^n) \to \text{Mob}(\mathbb{E}^{n+1}_+)$ is injective.

Clearly $\overline{\phi \circ \psi} = \bar{\phi} \circ \bar{\psi}$. Hence, $\phi \mapsto \bar{\phi} : \text{Mob}(\hat{\mathbb{E}}^n) \to \text{Mob}(\mathbb{E}^{n+1}_+)$ is a homomorphism.

It remains to prove $\phi \mapsto \bar{\phi} : \text{Mob}(\hat{\mathbb{E}}^n) \to \text{Mob}(\mathbb{E}^{n+1}_+)$ is surjective. Let $\psi \in \text{Mob}(\mathbb{E}^{n+1}_+)$. The continuity of ψ implies $\psi((\mathbb{E}^n \times \{0\}) \cup \infty) = (\mathbb{E}^n \times \{0\}) \cup \{\infty\}$. Define $\phi : \hat{\mathbb{E}}^n \to \hat{\mathbb{E}}^n$ by $\overline{\phi(x)} = \psi(\bar{x})$. For $x, y \in \mathbb{E}^n$, since $\|x - y\| = \|\bar{x} - \bar{y}\|$, then $[u, v, x, y] = [\bar{u}, \bar{v}, \bar{x}, \bar{y}]$ for $u, v, x, y \in \hat{\mathbb{E}}^n$. Since ψ is a Möbius transformation, it preserves cross-ratios by Theorem 4.14. Hence, for $u,v,x,y \in \hat{E}^n$,

$[\phi(u),\phi(v),\phi(x),\phi(y)] = [\overline{\phi(u)},\overline{\phi(v)},\overline{\phi(x)},\overline{\phi(y)}] =$
$[\psi(\bar{u}),\psi(\bar{v}),\psi(\bar{x}),\psi(\bar{y})] = [\bar{u},\bar{v},\bar{x},\bar{y}] = [u,v,x,y]$.

Thus ϕ preserves cross-ratios. Hence, $\phi \in \mathrm{Mob}(\hat{E}^n)$ by Theorem 4.14. It follows that $\bar{\phi} \in \mathrm{Mob}(E^{n+1}_+)$ from the first paragraph of this proof. Furthermore, for $x \in \hat{E}^n$, $\psi(\bar{x}) = \overline{\phi(x)} = \bar{\phi}(\bar{x})$. Hence, $\psi | E^n \times \{0\} = \bar{\phi} | E^n \times \{0\}$. Therefore, $\psi = \bar{\phi}$ by Corollary 4.20. $\square$

<u>Corollary 4.24</u> If $\phi \in \mathrm{Mob}(E^{n+1}_+)$, then $\phi = J_1 \circ \cdots \circ J_k$ where each $J_i \in \mathrm{Mob}(E^{n+1}_+)$, J_i is the extended reflection/inversion in $S_i \in \Sigma^{n+1}$, and S_i is orthogonal to $E^n \times \{0\}$, for $1 \le i \le k$.

<u>Proof</u> Theorem 4.23 implies $\phi = \bar{\psi}$ where $\psi \in \mathrm{Mob}(\hat{E}^n)$. Now $\psi = J_1 \circ \cdots \circ J_k$ where each J_i is the extended reflection/inversion in some $S_i \in \Sigma^n$ for $1 \le i \le k$. In the first paragraph of the previous proof, we observed that $\bar{\psi} = \bar{J}_1 \circ \cdots \circ \bar{J}_k$ where $\bar{J}_i$ is the extended reflection/inversion in $\bar{S}_i \in \Sigma^{n+1}$ and $\bar{S}_i$ is orthogonal to

$\mathbb{E}^n \times \{0\}$. Since $\overline{S}_i$ is orthogonal to $\mathbb{E}^n \times \{0\}$, then Theorem 4.9 implies $\overline{J}_i \in Mob(\mathbb{E}^{n+1}_+)$. Thus, $\phi = \overline{J}_1 \circ \cdots \circ \overline{J}_k$. $\square$

<u>Corollary 4.25.</u> If $S \in \Sigma^{n+1}$ and V is a component of $\hat{\mathbb{E}}^{n+1} - S$, then every $\phi \in Mob(V)$ is of the form $\phi = J_1 \circ \cdots \circ J_k$ where each $J_i \in Mob(V)$, J_i is the extended reflection/inversion in $T_i \in \Sigma^{n+1}$ and T_i is orthogonal to S for $1 \le i \le k$.

<u>Proof</u>. There is a $\psi \in Mob(\hat{\mathbb{E}}^{n+1})$ such that $\psi((\mathbb{E}^n \times \{0\}) \cup \{\infty\}) = S$ and $\psi(\mathbb{E}^{n+1}_+) = V$. Let $\phi \in Mob(V)$. Then $\psi^{-1} \circ \phi \circ \psi(\mathbb{E}^{n+1}_+) = \psi^{-1} \circ \phi(V) = \psi^{-1}(V) = \mathbb{E}^{n+1}_+$. So $\psi^{-1} \circ \phi \circ \psi \in Mob(\mathbb{E}^{n+1}_+)$. Hence, Corollary 4.24 implies $\psi^{-1} \circ \phi \circ \psi = J_1 \circ \cdots \circ J_k$ where $J_i \in Mob(\mathbb{E}^{n+1}_+)$, J_i is extended reflection/inversion in $T_i \in \Sigma^{n+1}$ and T_i is orthogonal to $\mathbb{E}^n \times \{0\}$, for $1 \le i \le k$. Hence,
$\phi = \psi \circ (\psi^{-1} \circ \phi \circ \psi) \circ \psi^{-1} = \psi \circ (J_1 \circ \cdots \circ J_k) \psi^{-1} = (\psi \circ J_1 \circ \psi^{-1}) \circ \cdots \circ (\psi \circ J_k \circ \psi^{-1})$. For $1 \le i \le k$, $\psi(T_i) \in \Sigma^{n+1}$, Corollary 4.17 implies $\psi \circ J_i \circ \psi^{-1}$ is the extended reflection inversion in $\psi(T_i)$, and $\psi(T_i)$ is orthogonal to $\psi(\mathbb{E}^n \times 0) = S$ because ψ is conformal. Also since
$\psi \circ J_i \circ \psi^{-1}(V) = \psi \circ J_i(\mathbb{E}^{n+1}_+) = \psi(\mathbb{E}^{n+1}_+) = V$,
then $\psi \circ J_i \circ \psi^{-1} \in Mob(V)$. $\square$

<u>Def</u> If $u_1, \ldots, u_k$ is an orthonormal sequence in $\mathbb{E}^n$ and $a_1, \ldots, a_k \in \mathbb{R}$, then call the set $P(u_1,a_1) \cap P(u_2,a_2) \cap \cdots \cap P(u_k, a_k)$ an <u>$(n-k)$-dimensional plane</u> in $\mathbb{E}^n$. If $u_1, \ldots, u_{k-1}$ is an orthonormal sequence in $\mathbb{E}^n$, $a_1, \ldots, a_{k-1} \in \mathbb{R}$, $c \in P(u_1,a_1) \cap \cdots \cap P(u_{k-1}, a_{k-1})$ and $r > 0$, then call the set $S(c,r) \cap P(u_1,a_1) \cap \cdots \cap P(u_{k-1},a_{k-1})$ an <u>$(n-k)$-dimensional sphere</u> in $\mathbb{E}^n$. For $1 \leq k \leq n$, let Σ_k^n denote the set of all $(n-k)$-dimensional planes and $(n-k)$-dimensional spheres in $\mathbb{E}^n$.

Observe that $\Sigma^n = \Sigma_1^n$, Σ_{n-1}^n is the set of all straight lines and circles in $\mathbb{E}^n$, and Σ_n^n is the set of all one- and two-point sets in $\mathbb{E}^n$.

<u>Lemma 4.26.</u> a) If $c, d \in \mathbb{E}^n$ and $r, s > 0$, then there is a unit vector $u \in \mathbb{E}^n$ and an $a \in \mathbb{R}$ such that $S(c,r) \cap S(d,s) = S(c,r) \cap P(u,a)$.

b) If $T = S(c_1,r_1) \cap \cdots \cap S(c_\ell, r_\ell) \cap P(v_1, b_1) \cap \cdots \cap P(v_m, b_m)$ is a non-empty subset of $\mathbb{E}^n$ where $\ell \geq 0$, $m \geq 1$, $c_1, \ldots, c_\ell \in \mathbb{E}^n$, $r_1, \ldots, r_\ell > 0$, $v_1, \ldots, v_m$ are unit vectors in $\mathbb{E}^n$ and $b_1, \ldots, b_m \in \mathbb{R}$, then $T \in \bigcup_{k=1}^n \Sigma_k^n$.

Homework Problem 4.2. Prove Lemma 4.26.

Theorem 4.27. a) If $\phi \in \mathrm{Mob}(\hat{\mathbb{E}}^n)$ and $S \in \Sigma^n_k$ where $1 \le k \le n$, then $\phi(S) \in \Sigma^n_k$.
b) $\mathrm{Mob}(\hat{\mathbb{E}}^n)$ acts transitively on Σ^n_k for $1 \le k \le n$.

Homework Problem 4.3. Prove Theorem 4.27.

We now generalize the definition of orthogonal to cover submanifolds of a Riemannian n-manifold that are of dimension less than $n-1$.

Definition Let S and T be vector subspaces of a finite dimensional vector space V. Let

$$S+T = \{x+y : x \in S \text{ and } y \in T\}.$$

Observe that $S+T$ is a vector subspace of V. S and T are maximally independent if

$$\dim(S+T) = \min\{\dim(S) + \dim(T), \dim(V)\}$$

Since $\dim(S+T) = \dim(S) + \dim(T) - \dim(S \cap T)$, then S and T are maximal dimensional if and only if

$$\dim(S \cap T) = \max\{0, \dim(S) + \dim(T) - \dim(V)\}.$$

<u>Definition</u> Suppose S is a vector subspace of an inner product space V. Recall that $S^\perp = \{x \in V : x \cdot y = 0 \text{ for all } y \in S\}$ and $S^\perp$ is a vector subspace of V. For $A \subseteq S$, define

$$A^\perp_S = \{x \in S : x \cdot y = 0 \text{ for all } y \in A\}.$$

Then $A^\perp_S$ is a vector subspace of S.

<u>Definition</u> Let S and T be vector subspaces of a finite dimensional inner product space V. Then S <u>is orthogonal to</u> T if S and T are maximally independent and any one of the following three equivalent conditions holds.

- $(S \cap T)^\perp_S \subset T^\perp_V$.
- $(S \cap T)^\perp_T \subset S^\perp_V$
- For all $x \in (S \cap T)^\perp_S$ and $y \in (S \cap T)^\perp_T$, $x \cdot y = 0$.

Lemma 4.28. If S and T are vector subspaces of a finite dimensional inner product space V, then the following three statements are equivalent.

a) $(S\cap T)^{\perp}_{S} \subset T^{\perp}_{V}$

b) $(S\cap T)^{\perp}_{T} \subset S^{\perp}_{V}$

c) For all $x \in (S\cap T)^{\perp}_{S}$ and $y \in (S\cap T)^{\perp}_{T}$, $x \cdot y = 0$.

Furthermore, if $S+T=V$ (which is equivalent to the statement: S and T are maximally independent and $\dim(S)+\dim(T) \geq \dim(V)$), then a) is equivalent to $(S\cap T)^{\perp}_{S} = T^{\perp}_{V}$, and b) is equivalent to $(S\cap T)^{\perp}_{T} = S^{\perp}_{V}$.

Homework Problem 4.4. Prove Lemma 4.28.

Lemma 4.29 Suppose R, S and T are vector subspaces of a finite dimensional inner product space V. Then:

a) $S^{\perp\perp} = S$,

b) $(S\cap T)^{\perp} = S^{\perp} + T^{\perp}$

c) If R is orthogonal to S and R is orthogonal to T, then R is orthogonal to $S\cap T$.

Homework Problem 4.5. Prove Lemma 4.29.

<u>Def</u> Suppose L and M are differentiable submanifolds of a Riemannian manifold N. For $x \in L \cap M$, L <u>is orthogonal to</u> M <u>at</u> x if $T_x(L)$ is orthogonal to $T_x(M)$ in $T_x(N)$. L <u>is orthogonal to</u> M if $L \cap M \neq \emptyset$ and L is orthogonal to M at every point of $L \cap M$.

<u>Lemma 4.30.</u> Suppose L and M are differentiable submanifolds of a Riemannian manifold N, and suppose $\varphi: N \to N'$ is a conformal diffeomorphism from N to a Riemannian manifold N'. If L is orthogonal to M at point $x \in L \cap M$, then $\varphi(L)$ is orthogonal to $\varphi(M)$ at $\varphi(x)$. Hence, if L is orthogonal to M, then $\varphi(L)$ is orthogonal to $\varphi(M)$.

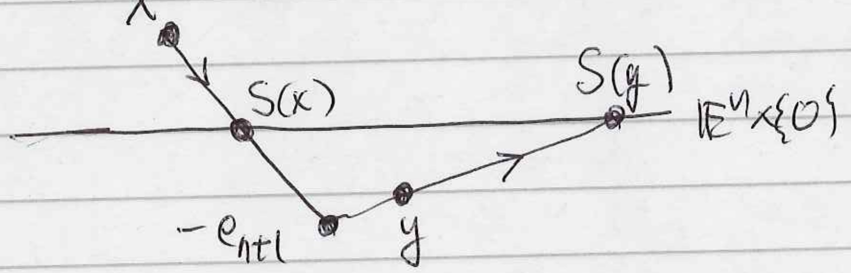
<u>Homework Problem 4.6.</u> Prove Lemma 4.30.

<u>Definition</u> <u>Stereographic projection from $-e_{n+1}$</u> is the map

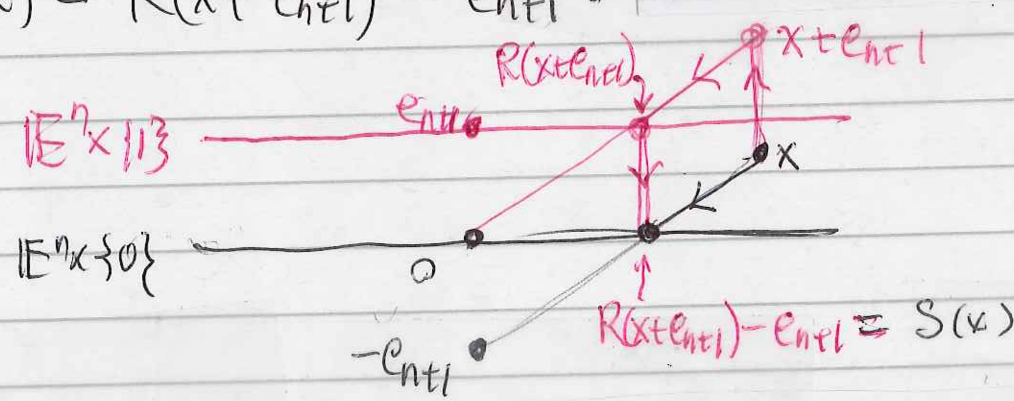
$$S : \mathbb{E}^n \times (-1, \infty) \to \mathbb{E}^n \times \{0\}$$

determined by the formula

$$S(x) = \frac{x - x_{n+1} e_{n+1}}{x_{n+1} + 1}$$



Recall that radial projection is the map $R : \mathbb{E}^n \times (0, \infty) \to \mathbb{E}^n \times \{1\}$ determined by the formula $R(x) = (1/x_{n+1})x$. Observe that $S(x) = R(x + e_{n+1}) - e_{n+1}$. We can verify



this observation by computation: $R(x+e_{n+1}) - e_{n+1} =$

$$\frac{x+e_{n+1}}{x_{n+1}+1} - e_{n+1} = \frac{x + e_{n+1} - (x_{n+1}+1)e_{n+1}}{x_{n+1}+1} = \frac{x - x_{n+1}e_{n+1}}{x_{n+1}+1} = S(x).$$

Observe that $S(x)\cdot e_{n+1} = \dfrac{x_{n+1}-x_{n+1}}{x_{n+1}+1} = 0$.

Hence, $S(\mathbb{E}^n \times (-1,\infty)) \subset \mathbb{E}^n \times \{0\}$.

Regard $\mathbb{E}^n \times (-1,\infty)$ as a subset of $\mathbb{M}^{n+1}$. Then $\mathbb{H}^n \subset \mathbb{E}^n \times (-1,\infty)$. Define $\sigma : \mathbb{H}^n \to \mathbb{E}^n \times \{0\}$ by $\sigma = S|\mathbb{H}^n$.

Recall $U^n = \{x \in \mathbb{E}^n : \|x\| < 1\}$. Define $\tau : U^n \times \{0\} \to \mathbb{M}^{n+1}$ by

$$\tau(y) = \left(\frac{2}{1-\|y\|^2}\right)y + \left(\frac{1+\|y\|^2}{1-\|y\|^2}\right)e_{n+1}.$$

<u>Lemma 4.31</u>. $\sigma : \mathbb{H}^n \to (U^n \times \{0\})$ is a conformal diffeomorphism and $\sigma^{-1} = \tau$.

<u>Proof</u> First we show $\sigma(\mathbb{H}^n) \subset U^n \times \{0\}$. $\sigma(\mathbb{H}^n) \subset \mathbb{E}^n \times \{0\}$. Let $x \in \mathbb{H}^n$. Then

$$\|\sigma(x)\|^2 = \sigma(x)\cdot\sigma(x) = \sigma(x)\circ\sigma(x) = \left(\frac{x - x_{n+1}e_{n+1}}{x_{n+1}+1}\right)\circ\left(\frac{x - x_{n+1}e_{n+1}}{x_{n+1}+1}\right) = \frac{x\circ x - 2x_{n+1}(x\circ e_{n+1}) + x_{n+1}^2(e_{n+1}\circ e_{n+1})}{(x_{n+1}+1)^2} =$$

$$\frac{-1+2x_{n+1}^2 - x_{n+1}^2}{(x_{n+1}+1)^2} = \frac{x_{n+1}^2-1}{(x_{n+1}+1)^2} = \frac{x_{n+1}-1}{x_{n+1}+1}.$$

Since $x_{n+1} \geq 1$, then $0 \leq x_{n+1}-1 < x_{n+1}+1$.

Hence $\|\sigma(x)\| < 1$. It follows that $\sigma(\mathbb{H}^n) \subset U^n \times \{0\}$.

Next we show that $\tau(U^n \times \{0\}) \subset \mathbb{H}^n$.
Let $y \in U^n \times \{0\}$. Then

$$\tau(y) \cdot e_{n+1} = \frac{1+\|y\|^2}{1-\|y\|^2} \geq 1$$

So $\tau(U^n \times \{0\}) \subset E^n \times [1, \infty)$.
Again let $y \in U^n \times \{0\}$. Then

$$\tau(y) \circ \tau(y) = \left(\frac{2}{1-\|y\|^2}\right)^2 \|y\|^2 + \left(\frac{1+\|y\|^2}{1-\|y\|^2}\right)^2 (-1) =$$

$$\frac{4\|y\|^2 - 1 - 2\|y\|^2 - \|y\|^4}{(1-\|y\|^2)^2} = -\frac{1-2\|y\|^2+\|y\|^4}{(1-\|y\|^2)^2} =$$

$$-\frac{(1-\|y\|^2)^2}{(1-\|y\|^2)^2} = -1.$$

It follows that $\tau(U^n \times \{0\}) \subset \mathbb{H}^n$.

Next we verify that $\tau \circ \sigma = id_{\mathbb{H}^n}$ and $\sigma \circ \tau = id_{U^n \times \{0\}}$.

Recall that $\|\sigma(x)\|^2 = \dfrac{x_{n+1}-1}{x_{n+1}+1}$. Hence,

$$\tau \circ \sigma(x) = \left(\frac{2}{1-\|\sigma(x)\|^2}\right)\sigma(x) + \left(\frac{1+\|\sigma(x)\|^2}{1-\|\sigma(x)\|^2}\right) e_{n+1} =$$

$$\left(\frac{2}{1-\left(\frac{x_{n+1}-1}{x_{n+1}+1}\right)}\right)\left(\frac{x - x_{n+1}e_{n+1}}{x_{n+1}+1}\right) + \left(\frac{1+\left(\frac{x_{n+1}-1}{x_{n+1}+1}\right)}{1-\left(\frac{x_{n+1}-1}{x_{n+1}+1}\right)}\right) e_{n+1} =$$

$$\left(\frac{2}{2}\right)(x - x_{n+1}e_{n+1}) + \left(\frac{2x_{n+1}}{2}\right)e_{n+1} =$$

$$x - x_{n+1}e_{n+1} + x_{n+1}e_{n+1} = x.$$

Thus, $\tau \circ \sigma = \mathrm{id}_{\mathbb{H}^n}$

Recall that $\tau(y) \cdot e_{n+1} = \frac{1+\|y\|^2}{1-\|y\|^2}$. Thus

$$\sigma \circ \tau(y) = \frac{\tau(y) - (\tau(y)\cdot e_{n+1})e_{n+1}}{(\tau(y)\cdot e_{n+1}) + 1} =$$

$$\frac{\tau(y) - \left(\frac{1+\|y\|^2}{1-\|y\|^2}\right)e_{n+1}}{\frac{1+\|y\|^2}{1-\|y\|^2} + 1} = \frac{(1-\|y\|^2)\tau(y) - (1+\|y\|^2)e_{n+1}}{2} =$$

$$\frac{2y + (1+\|y\|^2)e_{n+1} - (1+\|y\|^2)e_{n+1}}{2} = y.$$

Thus $\sigma \circ \tau = \mathrm{id}_{U^n \times \{0\}}$.

Thus $\sigma : \mathbb{H}^n \to U^n \times \{0\}$ is a diffeomorphism and $\sigma^{-1} = \tau$.

We now verify that σ is conformal. For $x \in \mathbb{H}^n$, $d\sigma_x = dS_x \mid T_x(\mathbb{H}^n)$. So we will calculate dS_x.

For $x \in \mathbb{E}^n \times (-1,\infty)$ and $v \in \mathbb{E}^{n+1}$,

$$dS_x(v) = \lim_{t\to 0} \frac{S(x+tv) - S(x)}{t} =$$

$$\lim_{t\to 0} \frac{1}{t}\left(\frac{(x+tv) - (x_{n+1}+tv_{n+1})e_{n+1}}{x_{n+1}+tv_{n+1}+1} - \frac{x - x_{n+1}e_{n+1}}{x_{n+1}+1} \right) =$$

$$\lim_{t\to 0} \frac{1}{t}\left(\frac{(x_{n+1}+1)\left((x+tv) - (x_{n+1}+tv_{n+1})e_{n+1}\right) - (x_{n+1}+tv_{n+1}+1)(x - x_{n+1}e_{n+1})}{(x_{n+1}+tv_{n+1}+1)(x_{n+1}+1)} \right)$$

$$= \lim_{t\to 0} \frac{1}{t}\left(\frac{(x_{n+1}+1)(tv - tv_{n+1}e_{n+1}) - tv_{n+1}(x - x_{n+1}e_{n+1})}{(x_{n+1}+tv_{n+1}+1)(x_{n+1}+1)} \right) =$$

$$\lim_{t\to 0} \frac{(x_{n+1}+1)(v - v_{n+1}e_{n+1}) - v_{n+1}(x - x_{n+1}e_{n+1})}{(x_{n+1}+tv_{n+1}+1)(x_{n+1}+1)} =$$

$$\frac{(x_{n+1}+1)v - (x_{n+1}+1)v_{n+1}e_{n+1} - v_{n+1}(x - x_{n+1}e_{n+1})}{(x_{n+1}+1)^2} =$$

$$\frac{(x_{n+1}+1)v - v_{n+1}(x + e_{n+1})}{(x_{n+1}+1)^2}$$

To prove σ is conformal, let $x \in H^n$ and let $v, w \in T_x(H^n)$. Then $x \circ x = -1$ and $x \circ v = 0 = x \circ w$. Observe that

$$dS_x(v)\cdot e_{n+1} = \frac{(x_{n+1}+1)v_{n+1} - v_{n+1}(x_{n+1}+1)}{(x_{n+1}+1)^2} = 0.$$

Hence, $d\sigma_x(v)\cdot d\sigma_x(w) = dS_x(v)\cdot dS_x(w) = dS_x(v)\circ dS_x(w) =$

$$\left(\frac{(x_{n+1}+1)v - v_{n+1}(x+e_{n+1})}{(x_{n+1}+1)^2}\right)\circ\left(\frac{(x_{n+1}+1)w - w_{n+1}(x+e_{n+1})}{(x_{n+1}+1)^2}\right) =$$

$$\frac{(x_{n+1}+1)^2(v\circ w) + 2(x_{n+1}+1)v_{n+1}w_{n+1} + v_{n+1}w_{n+1}(-1-2x_{n+1}-1)}{(x_{n+1}+1)^4} =$$

$$\frac{(x_{n+1}+1)^2(v\circ w) + 2(x_{n+1}+1)v_{n+1}w_{n+1} - 2(x_{n+1}+1)v_{n+1}w_{n+1}}{(x_{n+1}+1)^4} =$$

$$\frac{1}{(x_{n+1}+1)^2}(v\circ w).$$

Hence, Lemma 4.1 implies $d\sigma_x$ is angle preserving. Therefore σ is conformal. $\square$

<u>Definition</u> Define the metric η_{U} on $U^n \times \{0\}$ by

$$\eta_U(x,y) = \eta(\tau(x), \tau(y)).$$

Then $\eta(x,y) = \eta_U(\sigma(x), \sigma(y))$ and $\sigma : (\mathbb{H}^n, \eta) \to (U^n \times \{0\}, \eta_U)$ is an isometry.

We call $(U^n \times \{0\}, \eta_U)$ the <u>Poincaré ball model of $\mathbb{H}^n$</u>.

<u>Theorem 4.32.</u> For $x, y \in U^n \times \{0\}$,

$$\eta_U(x,y) = \cosh^{-1}\left(1 + \frac{2\,\|x-y\|^2}{(1-\|x\|^2)(1-\|y\|^2)}\right)$$

<u>Proof</u> Note that $x \circ y = x \cdot y$ and $x \circ e_{n+1} = 0 = y \circ e_{n+1}$.

Now $\cosh(\eta_U(x,y)) = \cosh(\eta(\tau(x), \tau(y))) =$

$$-\tau(x) \circ \tau(y) = -\left(\left(\frac{2}{1-\|x\|^2}\right)x + \left(\frac{1+\|x\|^2}{1-\|x\|^2}\right)e_{n+1}\right) \circ \left(\left(\frac{2}{1-\|y\|^2}\right)y + \left(\frac{1+\|y\|^2}{1-\|y\|^2}\right)e_{n+1}\right) =$$

$$\frac{-4\,x \circ y}{(1-\|x\|^2)(1-\|y\|^2)} - \frac{(1+\|x\|^2)(1+\|y\|^2)}{(1-\|x\|^2)(1-\|y\|^2)}(-1) =$$

$$\frac{1+\|x\|^2+\|y\|^2+\|x\|^2\|y\|^2-4x\cdot y}{(1-\|x\|^2)(1-\|y\|^2)} =$$

$$\frac{(1-\|x\|^2-\|y\|^2+\|x\|^2\|y\|^2)+2(\|x\|^2-2x\cdot y+\|y\|^2)}{(1-\|x\|^2)(1-\|y\|^2)} =$$

$$\frac{(1-\|x\|^2)(1-\|y\|^2)+2\|x-y\|^2}{(1-\|x\|^2)(1-\|y\|^2)} = 1+\frac{2\|x-y\|^2}{(1-\|x\|^2)(1-\|y\|^2)}.$$

Hence, $\eta_U(x,y) = \cosh^{-1}\left(1+\frac{2\|x-y\|^2}{(1-\|x\|^2)(1-\|y\|^2)}\right)$. $\square$

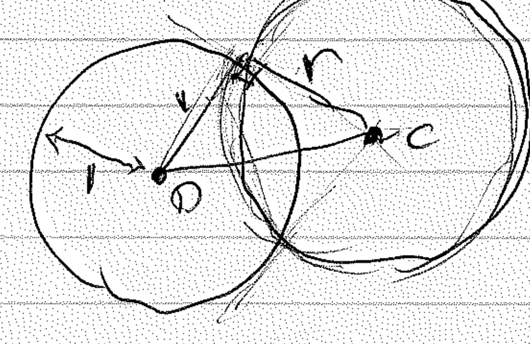
<u>Notation</u> We write $\mathbb{E}^n$, U^n and $\hat{\mathbb{E}}^n$ in place of $\mathbb{E}^n\times\{0\}$, $U^n\times\{0\}$ and $(\mathbb{E}^n\times\{0\})\cup\{\infty\}$ when no confusion can occur. This allows us to represent σ as a map from H^n to U^n, τ as a map from U^n to H^n, and η_U as a metric on U^n. In this way, we identify $\mathrm{Mob}(\hat{\mathbb{E}}^n)$ with $\mathrm{Mob}(\hat{\mathbb{E}}^n\times\{0\})$. Let $I_\eta(U^n)$ denote the isometry group of $U^n\ (=U^n\times\{0\})$ with respect to the metric η_U.

<u>Lemma 4.33</u>. For every $\phi\in\mathrm{Mob}(U^n)$, $\phi|U^n \in I_\eta(U^n)$.

Proof Corollary 4.25 implies that every Mobius transformation of U^n is the composition of extended reflections/inversions of the form J_T where $J_T \in Mob(U^n)$, $T \in \Sigma^n$ and T is orthogonal to $\partial U^n = S(0,1)$. So it suffices to prove each such J_T is an isometry of U^n with respect to η_U.

First suppose T is an extended hyperplane that is orthogonal to $S(0,1)$. Then Theorem 4.8.b implies $0 \in T$. Hence, $T = P(u,0)$ for some unit vector $u \in E^n$ and $J_T = \Sigma_{u,0}$. Thus, J_T is a Euclidean isometry that fixes 0. Therefore, $\|J_T(x)\| = \|J_T(x) - J_T(0)\| = \|x - 0\| = \|x\|$ and $\|J_T(x) - J_T(y)\| = \|x - y\|$. Hence, Theorem 4.32 implies $\eta_U(J_T(x), J_T(y)) = \eta_U(x,y)$. This proves $J_T|U^n \in \mathcal{I}_{\eta_U}(U^n)$.

Second suppose T is a hypersphere that is orthogonal to $S(0,1)$. Then Theorem 4.8.c implies $T = S(c,r)$ where $\|c\|^2 = 1 + r^2$, and



$J_T = \hat{I}_{cr}$. Lemma 4.12 implies

$$\|J_T(x) - J_T(y)\| = \frac{r^2 \|x-y\|}{\|x-c\| \, \|y-c\|}$$

Also

$$1 - \|J_T(x)\|^2 = 1 - \left\| \frac{r^2}{\|x-c\|^2}(x-c) + c \right\|^2 =$$

$$1 - \frac{r^4}{\|x-c\|^4}\|x-c\|^2 - \frac{2r^2}{\|x-c\|^2}((x-c)\cdot c) - \|c\|^2 =$$

$$\frac{\|x-c\|^2(1-\|c\|^2) - 2r^2((x-c)\cdot c) - r^4}{\|x-c\|^2} =$$

$$\frac{-r^2\|x-c\|^2 - 2r^2((x-c)\cdot c) - r^4}{\|x-c\|^2} \quad (\text{because } \|c\|^2 = 1+r^2) =$$

$$\frac{-r^2}{\|x-c\|^2}\left(\|x-c\|^2 + 2(x-c)\cdot c + r^2\right) =$$

$$\frac{-r^2}{\|x-c\|^2}\left(\|x-c\|^2 + 2(x-c)\cdot c + \|c\|^2 - 1\right) =$$

$$\frac{-r^2}{\|x-c\|^2}\left(\|(x-c)+c\|^2 - 1\right) = \frac{r^2}{\|x-c\|^2}\left(1 - \|x\|^2\right)$$

Thus,

$$\frac{\|J_T(x) - J_T(y)\|^2}{(1-\|J_X(x)\|^2)(1-\|J_T(y)\|^2)} = \frac{\left(\frac{r^4\,\|x-y\|^2}{\|x-c\|^2\,\|y-c\|^2}\right)}{\left(\frac{r^2}{\|x-c\|^2}(1-\|x\|^2)\right)\left(\frac{r^2}{\|y-c\|^2}(1-\|y\|^2)\right)}$$

$$= \frac{\|x-y\|^2}{(1-\|x\|^2)(1-\|y\|^2)}.$$

It follows from Theorem 4.3 that

$$\eta_{\mathcal{U}}(J_T(x), J_T(y)) = \eta_{\mathcal{U}}(x,y).$$

We conclude that $J_T|\mathcal{U}_n \in \mathcal{I}_{\eta_{\mathcal{U}}}(\mathcal{U}^n)$. $\square$

Def Let Γ^n denote the set of all hyperbolic hyperplanes in $\mathbb{H}^n$. In other words, Γ^n is the set of all subsets of $\mathbb{H}^n$ that are isometric to $\mathbb{H}^{n-1}$. The proof of Theorem 2.15 implies that the elements of Γ^n are precisely the non-empty subsets of $\mathbb{H}^n$ of the form $V \cap \mathbb{H}^n$ where V is an n-dimensional vector subspace of $\mathbb{M}^{n+1}$.

Def For $T \in \Sigma^n$, let $\Sigma^n[T]$ denote $\{S \in \Sigma^n : S$ is orthogonal to $T\}$.

We will prove:

Theorem 4.34 $T \mapsto \sigma(T)$ is a bijection from Γ^n to $\{S \cap U^n : S \in \Sigma^n[S^{n-1}]\}$.

Before proving this theorem, we state a closely related result. First we need some terminology.

Def If $f: X \to Y$ is an isometry between metric spaces, define the associated conjugation map $\chi_f : \mathcal{I}(X) \to \mathcal{I}(Y)$ by $\chi_f(g) = f \circ g \circ f^{-1}$ for $g \in \mathcal{I}(X)$.

Lemma 4.35 If $f: X \to Y$ is an isometry between metric spaces, then $\chi_f : \mathcal{I}(X) \to \mathcal{I}(Y)$ is an isomorphism.

Proof Since a composition of isometries is an isometry, then $\chi_f(g) \in I(Y)$ for each $g \in I(X)$. Thus χ_f maps $I(X)$ to $I(Y)$. It follows that $\chi_{f^{-1}}$ maps $I(Y)$ to $I(X)$. For $g, g' \in I(X)$,

$$\chi_f(g \circ g') = f \circ g \circ g' \circ f^{-1} = (f \circ g \circ f^{-1}) \circ (f \circ g' \circ f^{-1}) = \chi_f(g) \circ \chi_f(g').$$

Hence, $\chi_f : I(X) \to I(Y)$ is a group homomorphism. Therefore, $\chi_{f^{-1}} : I(Y) \to I(X)$ is also a group homomorphism. For $g \in I(X)$,

$$\chi_{f^{-1}} \circ \chi_f(g) = f^{-1}(f \circ g \circ f^{-1}) f = g.$$

Thus, $\chi_{f^{-1}} \circ \chi_f = id_{I(X)}$. Similarly $\chi_f \circ \chi_{f^{-1}} = id_{I(Y)}$. So $\chi_{f^{-1}} = (\chi_f)^{-1}$. Therefore, $\chi_f : I(X) \to I(Y)$ is an isomorphism. $\square$

Since $\sigma : H^n \to U^n \times \{0\}$ is an isometry (with respect to the metric η_u on $U^n \times \{0\}$), then Lemma 4.35 implies

$$\chi_\sigma : I(H^n) \to I_\eta(U^n)$$

is an isomorphism.

Def Let $Mob_1(U^n) = \{\phi | U^n : \phi \in Mob(U^n)\}$. Define the restriction map $\mathcal{R} : Mob(U^n) \to Mob_1(U^n)$ by $\mathcal{R}(\phi) = \phi | U^n$.

<u>Lemma 4.36</u>, $\mathcal{R}: \mathrm{Mob}(U^n) \to \mathrm{Mob}_1(U^n)$ is an isomorphism.

<u>Proof</u> $\mathcal{R}$ is clearly a surjective homomorphism. To prove injectivity, suppose $\phi \in \mathrm{Mob}(U^n)$ and $\mathcal{R}(\phi) = id_{U^n}$. Then $\phi | U^n = id_{\hat{\mathbb{E}}^n} | U^n$. So Corollary 4.20 implies $\phi = id_{\hat{\mathbb{E}}^n}$. □

Next we state a result that is closely related to Theorem 4.34.

<u>Theorem 4.37</u> $\mathrm{Mob}_1(U^n) = \mathcal{I}_\eta(U^n)$. Hence, $\mathcal{R}^{-1} \circ \chi_0 : \mathcal{I}(H^n) \to \mathrm{Mob}(U^n)$ is an isomorphism.

We will now prove Theorems 4.34 and 4.37. We will give two proofs of Theorem 4.34. The first proof provides more geometric insight but is more complicated. The second proof reduces the situation to an easily understood special case. We will also give two proofs of Theorem 4.37. The first proof reduces to a simple special case. The second proof invokes Theorem 4.34 to simplify the situation in a different way.

Proofs of Theorem 4.34

The two proofs of Theorem 4.34 begin with a lemma that deals with the simplest case of both proofs. We first recall some notation. If u is a unit vector in $\mathbb{M}^{n+1}$, then

$$P_u = \{x \in \mathbb{M}^{n+1} : x \circ u = 0\}$$

and

$$Z_u(x) = x - 2\left(\frac{x \circ u}{u \circ u}\right)u \quad \text{for } x \in \mathbb{M}^{n+1}.$$

If u is a unit vector in $\mathbb{E}^{n+1}$ and $a \in \mathbb{R}$, then

$$P(u,a) = \{x \in \mathbb{E}^{n+1} : x \cdot u = a\}$$

and

$$Z_{u,a}(x) = x - 2(x \cdot u - a)u \quad \text{for } x \in \mathbb{M}^{n+1}.$$

Lemma 4.38. Regard $\mathbb{E}^n \times \{0\}$ as a subset of both $\mathbb{M}^{n+1}$ and $\mathbb{E}^{n+1}$. If u is a unit vector in $\mathbb{E}^n \times \{0\}$, then $\sigma(P_u \cap \mathbb{H}^n) = P(u,0) \cap (U^n \times \{0\})$ and $\chi_\sigma(Z_u | \mathbb{H}^n) = Z_{u,0} | U^n \times \{0\}$.

Proof Since $u \in \mathbb{E}^n \times \{0\}$, then $x \circ u = x \cdot u$ for $x \in \mathbb{M}^{n+1}$.

Let $y \in \sigma(P_u \cap \mathbb{H}^n)$. Then $y = \sigma(x)$ where

$x \in P_u \cap \mathbb{H}^n$. Hence,

$$y \cdot u = \sigma(x) \cdot u = \frac{x \circ u - x_{n+1}(e_{n+1} \circ u)}{x_{n+1} + 1} = 0.$$

Thus, $y \in P(u,0) \cap (U^n \times \{0\})$.

On the other hand, if $y \in P(u,0) \cap (U^n \times \{0\})$, then $y = \sigma(x)$ for some $x \in \mathbb{H}^n$. Hence,

$$0 = y \cdot u = \frac{x \circ u - x_{n+1}(e_{n+1} \cdot u)}{x_{n+1} + 1} = \frac{x \circ u}{x_{n+1} + 1}.$$

Thus, $x \circ u = 0$. Therefore, $x \in P_u \cap \mathbb{H}^n$. So $y \in \sigma(P_u \cap \mathbb{H}^n)$.

We have proved $\sigma(P_u \cap \mathbb{H}^n) = P(u,0) \cap (U^n \times \{0\})$.

Since $x \circ u = x \cdot u$ for $x \in \mathbb{M}^{n+1}$ and ~~hence~~, $u \circ u$ ~~...~~ $= 1$, then $Z_u = Z_{u,0}$. Also since $u \cdot e_{n+1} = 0$, then $Z_{u0}(e_{n+1}) = e_{n+1}$. Furthermore, $Z_{u0}(x) \circ e_{n+1} = x \cdot e_{n+1} - 2(x \cdot u)(u \cdot e_{n+1}) = x_{n+1}$ for $x \in \mathbb{M}^{n+1}$. Since $Z_{u,0}$ is linear, then for $x \in \mathbb{H}^n$:

$$\sigma \circ Z_u(x) = \frac{Z_u(x) - (Z_u(x) \cdot e_{n+1})e_{n+1}}{Z_u(x) \cdot e_{n+1} + 1} = \frac{Z_{u0}(x) - x_{n+1} Z_{u0}(e_{n+1})}{x_{n+1} + 1}$$

$$= Z_{u,0}\left(\frac{x - x_{n+1} e_{n+1}}{x_{n+1} + 1}\right) = Z_{u0} \circ \sigma(x).$$

Thus, $\sigma \circ Z_u | \mathbb{H}^n = Z_{u,0} \circ \sigma$. Therefore,

$$\chi_\sigma(Z_u | \mathbb{H}^n) = \sigma \circ Z_u \circ \sigma^{-1} = Z_{u,0} | U^n \times \{0\}. \square$$

Observe that Lemma 4.38 establishes a portion of Theorem 4.34. If an element of Γ^n is of the form $P_u \cap \mathbb{H}^n$ where $u \in \mathbb{E}^n \times \{0\}$, then $\sigma(P_u \cap \mathbb{H}^n) = P(u0) \cap (U^n \times \{0\})$ where $P(u0) \cap (\mathbb{E}^n \times \{0\})$ is orthogonal to $S^{n-1} \times \{0\}$ because $0 \in P(u0)$.

<u>The first proof of Theorem 4.34</u>

We need:

<u>Lemma 4.39.</u> Let $\rho = R | \mathbb{H}^n : \mathbb{H}^n \to U^n \times \{1\}$. (Recall $R : \mathbb{E}^n \times (0, \infty) \to \mathbb{E}^n \times \{1\}$ is the radial retraction $R(x) = (1/x_{n+1})x$.) Let $S(0,1)_+ = S(0,1) \cap \mathbb{E}^{n+1}_+$ and define $\omega : U^n \times \{1\} \to S(0,1)_+$ by

$$\omega(x,1) = (x, \sqrt{1 - \|x\|^2})$$

for $x \in U^n$. Let $J = \hat{I}_{-e_{n+1}, \sqrt{2}} : \hat{\mathbb{E}}^{n+1} \to \hat{\mathbb{E}}^{n+1}$. Then $J(S(0,1)) = (\mathbb{E}^n \times \{0\}) \cup \{\infty\}$, $J(S(0,1)_+) = U^n \times \{0\}$ and

$$\sigma = J \circ \omega \circ \rho$$

Proof First we verify that $J(S(0,1)) = (\mathbb{E}^n \times \{0\}) \cup \{\infty\}$ and $J(S(0,1)_+) = U^n \times \{0\}$.

Let $x \in S(0,1) - \{-e_{n+1}\}$. Then $x_{n+1} > -1$ and
$\|x + e_{n+1}\|^2 = \|x\|^2 + 2x_{n+1} + 1 = 2(x_{n+1}+1) > 0$.

Hence,
$$J(x) = \frac{2(x+e_{n+1})}{\|x+e_{n+1}\|^2} - e_{n+1} = \frac{x+e_{n+1}}{x_{n+1}+1} - e_{n+1} = \frac{x - x_{n+1}e_{n+1}}{x_{n+1}+1}.$$

Therefore, $J(x) \circ e_{n+1} = 0$. Also $J(-e_{n+1}) = \infty$.
It follows that $J(S(0,1)) \subset (\mathbb{E}^n \times \{0\}) \cup \{\infty\}$.

Next let $x \in \mathbb{E}^n \times \{0\}$. Then
$\|x + e_{n+1}\|^2 = \|x\|^2 + 1$. Thus,
$$J(x) = \left(\frac{2}{\|x\|^2+1}\right)(x + e_{n+1}) - e_{n+1}.$$

Therefore,
$$\|J(x)\|^2 = \frac{4(\|x\|^2+1)}{(\|x\|^2+1)^2} - \frac{4}{\|x\|^2+1} + 1 = 1.$$

Also $J(\infty) = -e_{n+1}$. This proves
$J((\mathbb{E}^n \times \{0\}) \cup \{\infty\}) \subset S(0,1)$. Therefore,
$(\mathbb{E}^n \times \{0\}) \cup \{\infty\} = J \circ J((\mathbb{E}^n \times \{0\}) \cup \{\infty\}) \subset J(S(0,1))$.

We conclude that $J(S(0,1)) = (\mathbb{E}^n \times \{0\}) \cup \{\infty\}$.

Now let $x \in S(0,1)_+$. Then $0 < x_{n+1} \leq 1$. So

$$\|J(x)\|^2 = \frac{\|x - x_{n+1} e_{n+1}\|^2}{(x_{n+1}+1)^2} = \frac{\|x\|^2 - 2x_{n+1}^2 + x_{n+1}^2}{(x_{n+1}+1)^2} =$$

$$\frac{1 - x_{n+1}^2}{(1+x_{n+1})^2} = \frac{1 - x_{n+1}}{1 + x_{n+1}} < 1.$$

Consequently, $J(S(0,1)_+) \subseteq U^n \times \{0\}$.

Finally, let $x \in U^n \times \{0\}$. Then

$$J(x) \cdot e_{n+1} = \frac{2}{\|x\|^2+1} - 1 = \frac{1 - \|x\|^2}{1 + \|x\|^2} > 0.$$

Hence, $J(U^n \times \{0\}) \subseteq S(0,1)_+$. Consequently,

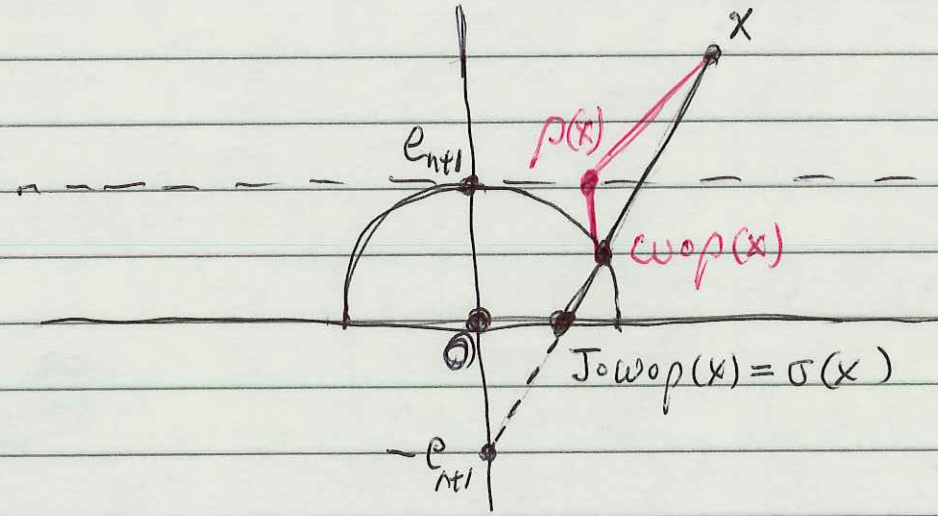
$U^n \times \{0\} = J \circ J(U^n \times \{0\}) \subset J(S(0,1)_+)$.

We conclude that $J(S(0,1)_+) = U^n \times \{0\}$.

Now let $x \in H^n$. Then

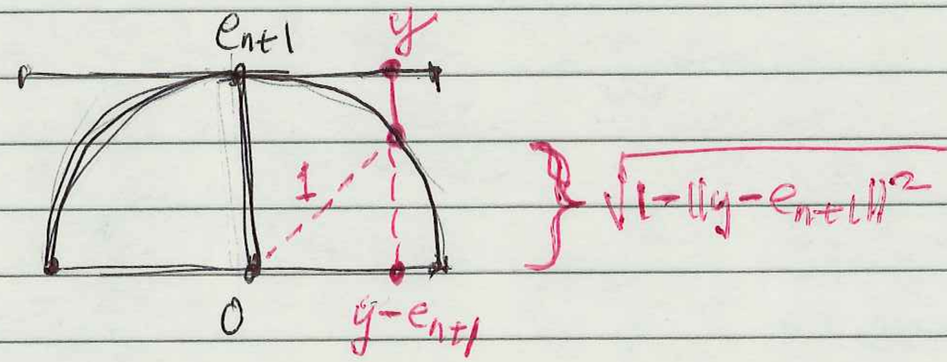
$$-1 = x \circ x = \|x\|^2 - 2x_{n+1}^2.$$

We know $\rho(x) = \frac{x}{x_{n+1}} \in U^n \times \{1\}$.



For $y \in U^n \times \{1\}$,

$$\omega(y) = y - e_{n+1} + \sqrt{1 - \|y - e_{n+1}\|^2}\, e_{n+1} \in S(0,1)_+ .$$



Also, if $y \in U^n \times \{1\}$, then

$$1 - \|y - e_{n+1}\|^2 = 1 - (\|y\|^2 - 2y \cdot e_{n+1} + \|e_{n+1}\|^2)$$
$$= 1 - (\|y\|^2 - 2 + 1) = 2 - \|y\|^2 .$$

Thus, $\omega(y) = y - e_{n+1} + \sqrt{2 - \|y\|^2}\, e_{n+1}$.

Therefore,

$$\mathrm{wop}(x) = \frac{x}{x_{n+1}} - e_{n+1} + \sqrt{2 - \frac{\|x\|^2}{x_{n+1}^2}}\, e_{n+1} =$$

$$\frac{x}{x_{n+1}} - e_{n+1} + \frac{\sqrt{2x_{n+1}^2 - \|x\|^2}}{x_{n+1}}\, e_{n+1} =$$

$$\frac{x}{x_{n+1}} - e_{n+1} + \frac{e_{n+1}}{x_{n+1}} = \frac{x + e_{n+1}}{x_{n+1}} - e_{n+1}.$$

Observe that

$$\|x + e_{n+1}\|^2 = (\|x\|^2 + 1) + 2x_{n+1} =$$

$$2x_{n+1}^2 + 2x_{n+1} = 2x_{n+1}(x_{n+1} + 1)$$

Hence,

$$J \circ \mathrm{wop}(x) = J\left(\frac{x + e_{n+1}}{x_{n+1}} - e_{n+1}\right) =$$

$$\frac{2}{\left\|\frac{x + e_{n+1}}{x_{n+1}}\right\|^2}\left(\frac{x + e_{n+1}}{x_{n+1}}\right) - e_{n+1} =$$

$$\left(\frac{2x_{n+1}^2}{2x_{n+1}(x_{n+1} + 1)}\right)\left(\frac{x + e_{n+1}}{x_{n+1}}\right) - e_{n+1} =$$

$$\frac{x + e_{n+1}}{x_{n+1} + 1} - e_{n+1} = \frac{x - x_{n+1}e_{n+1}}{x_{n+1} + 1} = \sigma(x). \ \square$$

We now begin the proof of Theorem 4.34. Let $T \in \Gamma^n$. We will find an $S \in \Sigma^n[S^{n-1}]$ such that $\sigma(T) = S \cap U^n$.

Theorem 2.15 implies $T = V \cap \mathbb{H}^n$ where V is an n-dimensional vector subspace of $\mathbb{M}^{n+1}$ such that $V \cap \mathbb{H}^n \neq \emptyset$.

First consider the case in which $e_{n+1} \in T$. Extend e_{n+1} to an orthonormal basis $u_2, \dots, u_n, e_{n+1}$ for V. Then extend $u_2, \dots, u_n, e_{n+1}$ to an orthonormal basis $u, u_2, \dots, u_n, e_{n+1}$ for $\mathbb{M}^{n+1}$. Since $u \circ e_{n+1} = 0$, then $u \in \mathbb{E}^n \times \{0\}$ and $V = P_u$. Observe that (after identifying $\mathbb{E}^n \times \{0\}$ with $\mathbb{E}^n$), $P(u,0) \cap (\mathbb{E}^n \times \{0\}) \in \Sigma^n[S^{n-1}]$. Also Lemma 4.38 implies

$$\sigma(T) = \sigma(P_u \cap \mathbb{H}^n) = P(u,0) \cap (U^n \times \{0\}).$$

This finishes the case in which $e_{n+1} \in T$.

Now for the harder case in which $e_{n+1} \notin T$. Choose $u_{n+1} \in T$ and extend u_{n+1} to an orthonormal basis $u_2, \dots, u_n, u_{n+1}$ for V. Then extend $u_2, \dots, u_{n+1}$ to an orthonormal basis $u, u_2, \dots, u_{n+1}$ for $\mathbb{M}^{n+1}$. Since $u_{n+1} \circ u_{n+1} = -1$, then $u \circ u = +1$ and $V = P_u$. (Also by replacing u_i by $-u_i$ (or u by $-u$) if necessary, we can assume $u_i \circ e_{n+1} > 0$ (and $u \circ e_{n+1} > 0$).

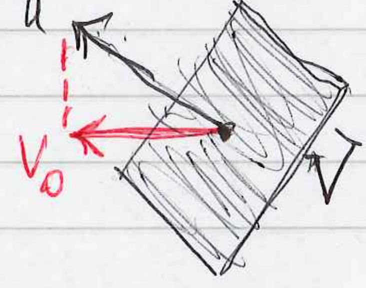
We assert $\rho(T) = V \cap (U^n \times \{1\})$.

<u>Proof</u>: Since $\rho(T) \subset R(V) \subset V$ and $\rho(T) \subset \rho(H^n) = U^n \times \{1\}$, then $\rho(T) \subset V \cap (U^n \times \{1\})$. On the other hand, if $y \in V \cap (U^n \times \{1\})$, then $y = \rho(x)$ for some $x \in H^n$. Thus, $y = (1/x_{n+1}) x$. So $x = x_{n+1} y \in V$. Thus, $x \in V \cap H^n = T$. Hence, $y = \rho(x) \in \rho(T)$. This proves $V \cap (U^n \times \{1\}) \subset \rho(T)$. $\square$

We assert there is a unit vector $v \in E^n \times \{0\}$ and an $a \in \mathbb{R}$ such that

$$\rho(T) = P(v, a) \cap (U^n \times \{1\}).$$

<u>Proof</u>. Let $v_0 = u + (u \circ e_{n+1}) e_{n+1}$. Then $v_0 \circ e_{n+1} = 0$.

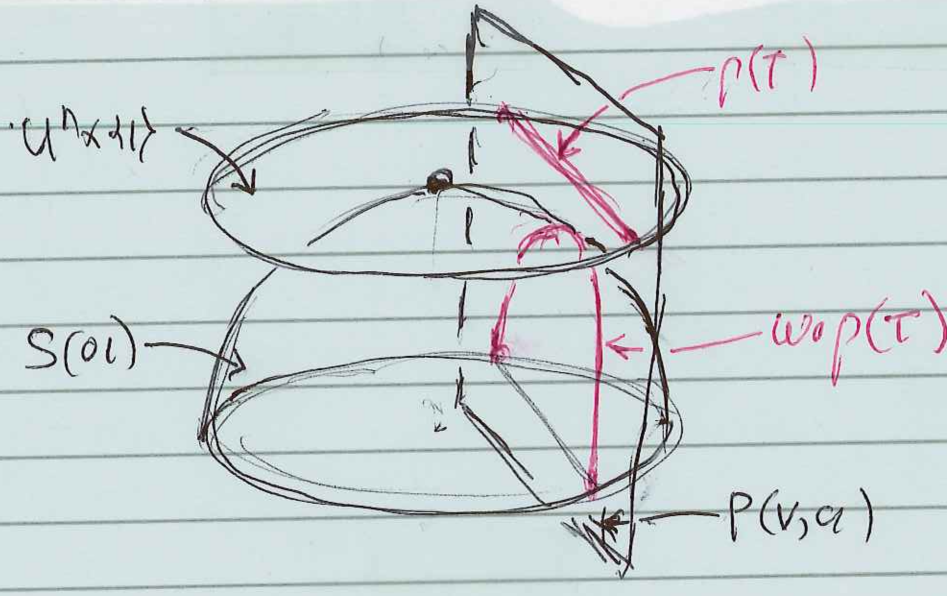


$v_0 \neq 0$ because: $v_0 = 0 \Rightarrow u = -(u \circ e_{n+1}) e_{n+1} \Rightarrow +1 = u \circ u = (u \circ e_{n+1})^2 (-1) \leq 0$, a contradiction.

Let $v = v_0/\|v_0\|$. Then v is a unit vector in $\mathbb{E}^n \times \{0\}$. Let $a = -(u \circ e_{n+1})/\|v_0\|$.

To prove $\rho(T) = P(v,a) \cap (U^n \times \{1\})$, it suffices to prove
$$P_u \cap (U^n \times \{1\}) = P(v,a) \cap (U^n \times \{1\}).$$
First let $x \in P_u \cap (U^n \times \{1\})$. Then $x \circ u = 0$ and $x \circ e_{n+1} = -1$. Hence,
$$x \cdot v = x \circ v = (x \circ v_0)/\|v_0\| =$$
$$(x \circ u + (u \circ e_{n+1})(x \circ e_{n+1}))/\|v_0\| = -(u \circ e_{n+1})/\|v_0\| = a.$$
Therefore, $x \in P(v,a) \cap (U^n \times \{1\})$. Second, suppose $x \in P(v,a) \cap (U^n \times \{1\})$. Then
$$x \circ v = x \cdot v = a \text{ and } x \circ e_{n+1} = -1.$$
Hence,
$$x \circ v_0 = x \circ (\|v_0\| v) = \|v_0\| a = -(u \circ e_{n+1}).$$
Since $v_0 = u + (u \circ e_{n+1}) e_{n+1}$, then
$$x \circ u = x \circ (v_0 - (u \circ e_{n+1}) e_{n+1}) = x \circ v_0 - (u \circ e_{n+1})(x \circ e_{n+1})$$
$$= -(u \circ e_{n+1}) - (u \circ e_{n+1})(-1) = 0.$$
Thus, $x \in P_u \cap (U^n \times \{1\})$.

We assert that $\omega \circ \rho(T) = P(V,a) \cap S(0,1)_+$

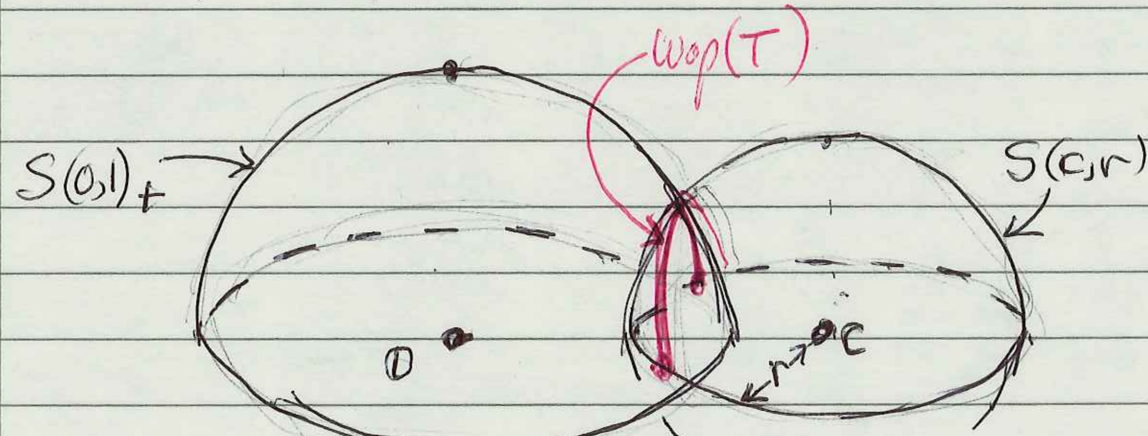


<u>Proof</u>: First suppose $z \in \omega \circ \rho(T)$. Then there is a $y \in \rho(T)$ such that $\omega(y) = z$. So $y \in P(V,a) \cap (U^n \times \{1\})$. We can write $y = (x,1)$ where $x \in U^n$. Hence, $(x,1) \cdot V = a$. Also $z = \omega(x,1) = (x, \sqrt{1-\|x\|^2})$. Since $V \in \mathbb{E}^n \times \{0\}$, then $(x,1) \cdot V = (x, \sqrt{1-\|x\|^2}) \cdot V$. Thus, $z \cdot V = (x,1) \cdot V = a$. Therefore, $z \in P(V,a)$. Also $\|z\|^2 = \|x\|^2 + (1-\|x\|^2) = 1$ and $\sqrt{1-\|x\|^2} > 0$. So $z \in S(0,1)_+$. This proves $z \in P(V,a) \cap S(0,1)_+$.

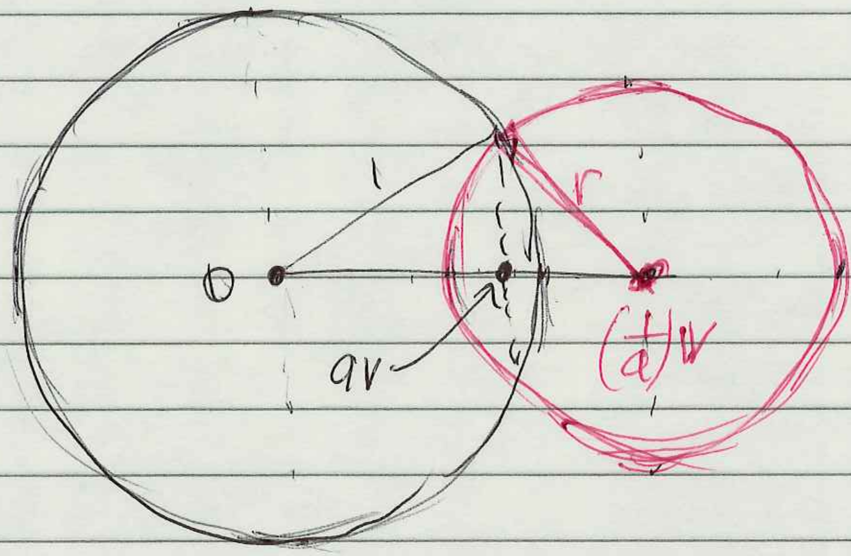
Second suppose $z \in P(V,a) \cap S(0,1)_+$. Then $z \cdot V = a$ and $z = (x,t)$ where $1 = \|z\|^2 = \|x\|^2 + t^2$ and $t > 0$. Thus, $x \in U^n$ and $z = (x, \sqrt{1-\|x\|^2})$. So $(x,1) \in U^n \times \{1\}$ and $\omega(x,1) = z$. Again since $V \in \mathbb{E}^n \times \{0\}$, then $(x,1) \cdot V = (x,t) \cdot V = a$. Thus, $(x,1) \in P(V,a) \cap (U^n \times \{1\}) = \rho(T)$ and $z = \omega(x,1)$. This proves $z \in \omega \circ \rho(T)$. $\square$

Since by hypothesis $e_{n+1} \notin T$, $\rho: \mathbb{H}^n \to U^n \times \{1\}$ is a bijection and $\rho(e_{n+1}) = e_{n+1}$, then $e_{n+1} \notin \rho(T) = P(v,a) \cap (U^n \times \{1\})$. Since $e_{n+1} \in U^n \times \{1\}$, then $e_{n+1} \notin P(v,a)$. Thus, $v \cdot e_{n+1} \neq a$. Since $v \in \mathbb{E}^n \times \{0\}$, then $v \cdot e_{n+1} = 0$. Consequently, $a \neq 0$.

We assert there is a $c \in \mathbb{E}^n \times \{0\}$ and an $r > 0$ such that $S(c,r)$ is orthogonal to $S(0,1)$ and $w \circ \rho(T) = S(c,r) \cap S(0,1)_+$.



Proof First we verify that $|a| < 1$.
Since $\rho(T) = P(v,a) \cap (U^n \times \{1\}) \neq \emptyset$, then there is an $x \in U^n$ such that $(x,1) \in P(v,a)$. Since $v \in \mathbb{E}^n \times \{0\}$, then $(x,1) \cdot v = (x,0) \cdot v$. Hence, $|a| = |(x,1)\cdot v| = |(x,0) \cdot v| \leq \|(x,0)\| \, \|v\| = \|x\| \leq 1$.



Let $c = \left(\frac{1}{a}\right)v$ and let $r = \sqrt{\frac{1}{a^2} - 1}$.

Then $c \in E^n \times 0$ and $\|c\|^2 = \frac{\|v\|^2}{a^2} = \frac{1}{a^2} = r^2 + 1$.

Therefore $S(0,1)$ and $S(c,r)$ are orthogonal.

To prove $wop(T) = S(c,r) \cap S(0,1)_+$, it suffices to prove

$$P(v,a) \cap S(0,1)_+ = S(c,r) \cap S(0,1)_+.$$

Suppose $x \in P(v,a) \cap S(0,1)_+$. Then $\|x\| = 1$ and $x \cdot v = a$. Hence

$$\|x - c\|^2 = \|x\|^2 - 2x \cdot c + \|c\|^2 = 1 - 2\left(\frac{x \cdot v}{a}\right) + \frac{\|v\|^2}{a^2} =$$

$$1 - 2\left(\frac{a}{a}\right) + \frac{1}{a^2} = \frac{1}{a^2} - 1 = r^2.$$

Thus, $x \in S(c,r)$. We have proved

$$P(v,a) \cap S(0,1)_+ \subset S(c,r) \cap S(0,1)_+.$$

Now suppose $x \in S(cr) \cap S(0,1)_+$. Then $\|x-c\| = r$ and $\|x\| = 1$. Hence,

$$\frac{1}{a^2} - 1 = r^2 = \|x-c\|^2 = \|x\|^2 - 2x\cdot c + \|c\|^2 =$$

$$1 - 2\frac{x\cdot v}{a} + \frac{\|v\|^2}{a^2} = 1 - 2\left(\frac{x\cdot v}{a}\right) + \frac{1}{a^2}.$$

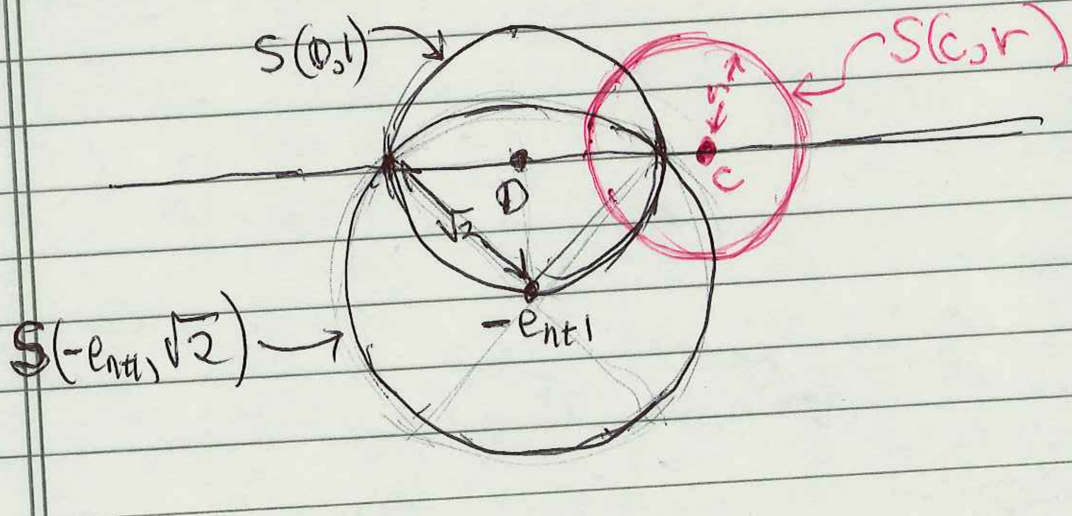
Thus $2\left(\frac{x\cdot v}{a}\right) = 2$. Therefore, $x\cdot v = a$. So $x \in P(v,a)$. We have proved

$$S(cr) \cap S(0,1)_+ \subset P(v,a) \cap S(0,1)_+ . \square$$

We assert that $S(cr)$ is orthogonal to $S(-e_{n+1}, \sqrt{2})$.

Proof: $\|c - (-e_{n+1})\|^2 = \|c\|^2 + 2c\cdot e_{n+1} + \|e_{n+1}\|^2 =$

$$\frac{\|v\|^2}{a^2} + 2\left(\frac{v\cdot e_{n+1}}{a}\right) + 1 = \frac{1}{a^2} + 1 = \left(\frac{1}{a^2} - 1\right) + 2 = r^2 + (\sqrt{2})^2. \square$$



Let $S = S(c,r) \cap (\mathbb{E}^n \times \{0\})$. Then S is a sphere in $\mathbb{E}^n \times \{0\}$ centered at c of radius r. S is orthogonal to $S(0,1) \cap (\mathbb{E}^n \times \{0\})$ ~~because~~ $= S^{n-1} \times \{0\}$ because

$$\|c - 0\|^2 = \frac{\|v\|^2}{a^2} = \frac{1}{a^2} = r^2 + 1.$$

Thus, if we identify $\mathbb{E}^n \times \{0\}$ with $\mathbb{E}^n$, then we can regard S as an element of Σ^n that is orthogonal to S^{n-1}.

We assert that $J \circ \omega \circ \rho(T) = S \cap (U^n \times \{0\})$.

Proof. $J \circ \omega \circ \rho(T) = J(S(c,r) \cap S(0,1)_+) = J(S(c,r)) \cap J(S(0,1)_+)$. Since $S(c,r)$ is orthogonal to $S(-e_{n+1}, \sqrt{2})$ and $J = \hat{I}_{-e_{n+1}, \sqrt{2}}$, then $J(S(c,r)) = S(c,r)$ by Corollary 4.21. Also $J(S(0,1)_+) = U^n \times \{0\}$ by Lemma 4.39.

Therefore,

$$J \circ \omega \circ \rho(T) = S(c,r) \cap (U^n \times \{0\}) = S \cap (U^n \times \{0\}). \quad \square$$

Since $\sigma = J \circ \omega \circ \rho$ by Lemma 4.39, then $\sigma(T) = S \cap (U^n \times \{0\})$.

Since $\sigma: \mathbb{H}^n \to (U^n \times \{0\})$ is a bijection, we conclude that the function $T \mapsto \sigma(T)$

maps Γ^n injectively into $\{S \cap U^n : S \in \Sigma^n[S^{n-1}]\}$.

To prove $T \mapsto \sigma(T)$ maps Γ^n onto $\{S \cap U^n : S \in \Sigma^n[S^{n-1}]\}$ we run the preceding arguments backwards.

Let $S \in \Sigma^n[S^{n-1}]$. First consider the case in which S is an ~~hyper~~ extended hyperplane in $\mathbb{E}^n$ that is orthogonal to S^{n-1}. Hence, $0 \in S$. By identifying $\mathbb{E}^n$ with $\mathbb{E}^n \times \{0\} \subset \mathbb{E}^{n+1}$, we have $S = P(u,0) \cap (\mathbb{E}^n \times \{0\})$ where u is a unit vector in $\mathbb{E}^n \times \{0\}$. In this case, $e_{n+1} \in P_u \cap \mathbb{H}^n$, $P_u \cap \mathbb{H}^n \in \Gamma^n$ and $\sigma(P_u \cap \mathbb{H}^n) = P(u,0) \cap (U^n \times \{0\}) = S \cap (U^n \times \{0\})$ by Lemma 4.38.

Second suppose S is a hypersphere in $\mathbb{E}^n$ that is orthogonal to S^{n-1}. Then there is a $c \in \mathbb{E}^n \times \{0\}$ and an $r > 0$ such that $S = S(c,r) \cap (\mathbb{E}^n \times \{0\})$ and $\|c\|^2 = 1 + r^2$.

We claim that $S(c,r)$ is orthogonal to $S(-e_{n+1}, \sqrt{2})$. Here's the proof:

$\|c - (-e_{n+1})\|^2 = \|c\|^2 + 2c \cdot e_{n+1} + \|e_{n+1}\|^2 =$

$(1 + r^2) + 0 + 1 = r^2 + (\sqrt{2})^2$.

It follows that $J(S(c,r)) = S(c,r)$. Also

$J(U^n \times \{0\}) = J \circ J(S(0,1)_+) = S(0,1)_+$ by Lemma 4.39. Thus,

$$J(S \cap (U^n \times \{0\})) = J(S(cr) \cap (U^n \times \{0\})) = S(cr) \cap S(01)_+.$$

Next we claim there is a unit vector $v \in E^n \times \{0\}$ and an $a \in \mathbb{R}$ such that $0 < a < 1$ and $S(cr) \cap (S(0,1)_+) = P(v,a) \cap S(0,1)_+$.

__Proof__. Let $v = (1/\|c\|)c$ and $a = 1/\|c\| = \frac{1}{\sqrt{1+r^2}}$.

If $x \in S(cr) \cap S(0,1)_+$, then $\|x-c\| = r$ and $\|x\| = 1$. Hence, $r^2 = \|x\|^2 - 2x\cdot c + \|c\|^2 = 1 - 2x\cdot c + 1 + r^2$. So $x \cdot c = 1$. Thus, $x \cdot v = x \cdot (c/\|c\|) = 1/\|c\| = a$. Therefore, $x \in P(va) \cap S(0,1)_+$.

On the other hand, if $x \in P(v,a) \cap S(0,1)_+$, then $x \cdot v = a$ and $\|x\| = 1$. Thus $x \cdot (c/\|c\|) = 1/\|c\|$. So $x \cdot c = 1$. Therefore,

$$\|x-c\|^2 = \|x\|^2 - 2x\cdot c + \|c\|^2 = 1 - 2 + (1+r^2) = r^2,$$

Hence, $x \in S(cr) \cap S(01)_+$.

We claim $\omega^{-1}(P(va) \cap S(01)_+) = P(va) \cap (U^n \times \{1\})$. For $(x,1) \in U^n \times \{1\}$, since $\omega(x,1) = (x, \sqrt{1-\|x\|^2})$ and $v \in E^n \times \{0\}$, then $(x,1)\cdot v = \omega(x,1)\cdot v$. Thus, $(x,1) \in P(va)$ if and only if $\omega(x1) \in P(va)$. Hence, the following statements are equivalent:

$(x,1) \in w^{-1}(P(va) \cap S(01)_+)$, $w(x1) \in P(va) \cap S(01)_+$, and $(x,1) \in P(va) \cap (U^n \times \{1\})$.

Next we assert there is a spacelike unit vector u in M^{n+1} such that $u_{n+1} > 0$ and $P_u \cap (U^n \times \{1\}) = P(va) \cap (U^n \times \{1\})$.

<u>Proof</u> Let $u = \dfrac{v + a e_{n+1}}{\sqrt{1-a^2}}$. Then

$$u \circ u = \frac{v \cdot v + 2a(v \circ e_{n+1}) + a^2(e_{n+1} \circ e_{n+1})}{1-a^2} =$$

$$\frac{1+0-a^2}{1-a^2} = 1,$$

Thus, u is a spacelike unit vector.

Also $u \cdot e_{n+1} = \dfrac{a}{\sqrt{1-a^2}} > 0$.

If $x \in P_u \cap (U^n \times \{1\})$, then $0 = x \circ u = \dfrac{x \cdot v - a}{\sqrt{1-a^2}}$.

Hence, $x \cdot v = a$. Thus, $x \in P(v,a) \cap (U^n \times \{1\})$.

If $x \in P(v,a) \cap (U^n \times \{1\})$, then

$$x \circ u = \frac{x \cdot v + a(x \circ e_{n+1})}{\sqrt{1-a^2}} = \frac{a-a}{\sqrt{1-a^2}} = 0.$$

Hence, $x \in P_u \cap (U^n \times \{1\})$.

Note that since $0 < a < 1$, then $av \in U^n \times \{0\}$. Hence, $av + e_{n+1} \in U^n \times \{1\}$. Since $(av + e_{n+1}) \cdot v = a$, then $av + e_{n+1} \in P(va) \cap (U^n \times \{1\}) = P_u \cap (U^n \times \{1\})$.

Thus, $P_u \cap (\mathcal{U}^n \times \{1\}) \neq \phi$. Since ρ maps $\mathbb{H}^n$ onto $\mathcal{U}^n \times \{1\}$, then there is an $x \in \mathbb{H}^n$ such that $\rho(x) \in P_u \cap (\mathcal{U}^n \times \{1\})$. Since $\rho(x) = x/x_{n+1}$, then $x/x_{n+1} \in P_u$. Therefore, $x \in P_u \cap \mathbb{H}^n$. Thus, $P_u \cap \mathbb{H}^n \neq \phi$. Let $T = P_u \cap \mathbb{H}^n$. Then $T \in \Gamma^n$. We proved earlier that in this situation $\rho(T) = P_u \cap (\mathcal{U}^n \times \{1\})$. Thus,

$$\rho(T) = P(va) \cap (\mathcal{U}^n \times \{1\}) = \omega^{-1}(P(va) \cap S(0\,1)_+).$$

Thus,

$$\omega \circ \rho(T) = P(va) \cap S(0\,1)_+ = S(c\,r) \cap S(0\,1)_+$$
$$= J(S \cap (\mathcal{U}^n \times \{0\})).$$

Consequently, by Lemma 4.39,

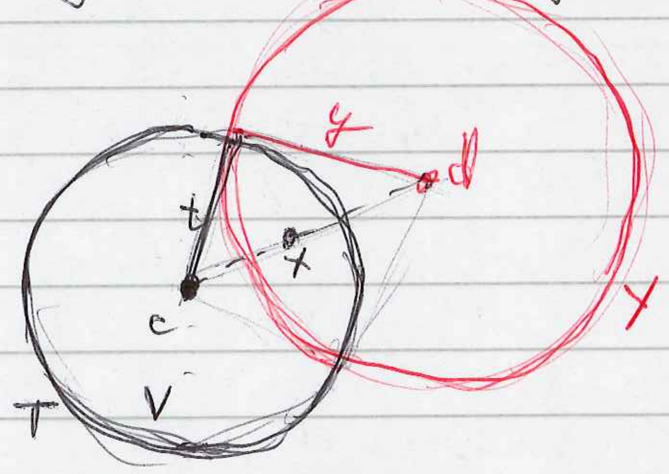
$$\sigma(T) = J \circ \omega \circ \rho(T) = J \circ J(S \cap (\mathcal{U}^n \times \{0\})) = S \cap (\mathcal{U}^n \times \{0\})$$

We conclude that $T \mapsto \sigma(T)$ is onto and, hence, a bijection from Γ^n to $\{S \cap \mathcal{U}^n : S \in \Sigma^n[S^{n-1}]\}$. $\square$

The second proof of Theorem 4.34.

We need a lemma which provides a Möbius transformation of $\mathcal{U}^n$ that moves a given point of $\mathcal{U}^n$ to $\mathbb{O}$. The following lemma accomplishes this with a single inversion.

Lemma 4.40. Let $T = S(c;t) \subset \mathbb{E}^n$ and let $V = \{x \in \mathbb{E}^n : \|x-c\| < t\}$. For every $x \in V - \{c\}$, there is a hypersphere $Y = S(d, y)$ which is orthogonal to T such that $J_Y \in Mob(V)$ and $J_Y(x) = c$. Specifically, $d = J_T(x)$ and $y = \sqrt{\|c-d\|^2 - t^2}$



Proof Since $J_T(x) = d$, then $\|x-c\|\|d-c\| = t^2$. Since $\|x-c\| < t$, then $\|d-c\| > t$. Thus, $y = \sqrt{\|c-d\|^2 - t^2}$ exists and is positive.

Also $t^2 + y^2 = \|c-d\|^2$. Hence, $T = S(c,\tau)$ and $Y = S(d,y)$ are orthogonal by Theorem 4.8.c. Thus, $J_Y \in Mob(V)$ and $J_T(Y) = Y$ by Corollary 4.21. Hence, Corollary 4.17 implies

$$J_Y = J_{J_T(Y)} = J_T \circ J_Y \circ (J_T)^{-1} = J_T \circ J_Y \circ J_T .$$

Thus,

$$J_Y(x) = J_T \circ J_Y \circ J_T(x) = J_T \circ J_Y(d) = J_T(\infty) = c. \quad \square$$

Suppose $T \in \Gamma^n$. First consider the case in which $e_{n+1} \in T$. We repeat the argument given in this situation in the first proof of Theorem 4.34 to obtain a unit vector $u \in \mathbb{E}^n \times \{0\}$ such that $P(u,0) \cap (\mathbb{E}^n \times \{0\}) \in \Sigma^n[S^{n-1}]$ and $\sigma(T) = P(u,0) \cap (\mathbb{U}^n \times \{0\})$. (We are identifying $\mathbb{E}^n \times \{0\}$ with $\mathbb{E}^n$.)

Now suppose $e_{n+1} \notin T$. Then $0 = \sigma(e_{n+1}) \notin \sigma(T)$. Choose a point $x \in \sigma(T)$. Lemma 4.40 provides a $\phi \in Mob(\mathbb{U}^n)$ such that $\phi(x) = 0$.

Lemma 4.33 implies that $\phi | U^n \in \mathcal{I}_\eta(U^n)$. Since $\sigma : \mathbb{H}^n \to U^n$ is an isometry, then it follows that $\sigma^{-1} \circ \phi \circ \sigma \in \mathcal{I}(\mathbb{H}^n)$. Since the elements of Γ^n are all the subsets of $\mathbb{H}^n$ that are isometric to $\mathbb{H}^{n-1}$, and $T \in \Gamma^n$, then $\sigma^{-1} \circ \phi \circ \sigma(T) \in \Gamma^n$. Since $0 = \phi(x) \in \phi \circ \sigma(T)$, then $e_{n+1} = \sigma^{-1}(0) \in \sigma^{-1} \circ \phi \circ \sigma(T)$. The argument in the preceding paragraph implies there is an $S \in \Sigma^n[S^{n-1}]$ such that $\sigma(\sigma^{-1} \circ \phi \circ \sigma(T)) = S \cap (U^n \times \{0\})$. Thus, $\phi \circ \sigma(T) = S \cap (U^n \times \{0\})$. Consequently,

$$\sigma(T) = \phi^{-1}(S) \cap \phi^{-1}(U^n \times \{0\})$$

Since $\phi \in Mob(U^n)$, then $\phi^{-1} \in Mob(U^n)$. Hence, $\phi^{-1}(U^n \times \{0\}) = U^n \times \{0\}$. Consequently, $\phi^{-1}(S^{n-1} \times \{0\}) = S^{n-1} \times \{0\}$. Since $S \in \Sigma^n$, then Theorem 4.5 implies $\phi^{-1}(S) \in \Sigma^n$. Since S is orthogonal to S^{n-1} and Mobius tranformations, being conformal, preserve orthogonality (by Lemma 4.7), then $\phi^{-1}(S)$ is orthogonal to $\phi^{-1}(S^{n-1}) = S^{n-1}$. Thus $\phi^{-1}(S) \in \Sigma^n[S^{n-1}]$.

We have now shown that $T \mapsto \sigma(T)$ maps Γ^n injectively into $\{S \cap U^n : S \in \Sigma^n[S^{n-1}]\}$.

We break the proof that $T \mapsto \sigma(T): \Gamma^n \to \{S \cap U^n : S \in \Sigma^n[S^{n-1}]\}$ is onto into two parts.

Let $S \in \Sigma^n[S^{n-1}]$ and suppose $0 \in S$. S can't be a hypersphere because if $S = S(c,r)$ where $\|c\|^2 = 1 + r^2$, then $0 \notin S$ because $\|0 - c\| = \|c\| > r$. Hence, $S = P(u,0) \cap (\mathbb{E}^n \times \{0\})$ where u is a unit vector in $\mathbb{E}^n \times \{0\}$. Therefore, $e_{n+1} \in P_u \cap H^n$, $P_u \cap H^n \in \Gamma^n$ and Lemma 4.38 implies

$$\sigma(P_u \cap H^n) = P(u,0) \cap (U^n \times \{0\}) = S \cap (U^n \times \{0\}).$$

Now suppose $S \in \Sigma^n[S^{n-1}]$ and $0 \notin S$. Choose $x \in S$. Lemma 4.40 provides a $\phi \in Mob(U^n)$ such that $\phi(x) = 0$. Since $S \in \Sigma^n$, then $\phi(S) \in \Sigma^n$ by Theorem 4.5. Since ϕ is conformal, then ϕ preserves orthogonality by Lemma 4.7. Therefore, since S is orthogonal to S^{n-1}, then $\phi(S)$ is orthogonal to $\phi(S^{n-1}) = S^{n-1}$. Thus, $\phi(S) \in \Sigma^n[S^{n-1}]$. Since $0 = \phi(x) \in \phi(S)$, then the preceding paragraph shows there is a $T \in \Gamma^n$ such that $\sigma(T) = \phi(S) \cap U^n$. (Here, we are identifying $\mathbb{E}^n$ with $\mathbb{E}^n \times \{0\}$.)

Since $\phi \in Mob(U^n)$, then $\phi^{-1} \in Mob(U^n)$. Therefore, $\phi^{-1}|U \in \mathcal{I}_\eta(U^n)$ by Lemma 4.33. Thus, $\sigma^{-1} \circ \phi^{-1} \circ \sigma \in \mathcal{I}_{H^n}$. Hence, $\sigma^{-1} \circ \phi^{-1} \circ \sigma(T) \in \Gamma^n$. Also

$$\sigma(\sigma^{-1} \circ \phi^{-1} \circ \sigma(T)) = \phi^{-1}(\sigma(T)) =$$

$$\phi^{-1}(\phi(S) \cap U^n) = S \cap \phi^{-1}(U^n) = S \cap U^n.$$

This completes the proof that $T \mapsto \sigma(T)$ maps Γ^n onto $\{S \cap U^n : S \in \Sigma^n[S^{n-1}]\}$. $\square$

First proof of Theorem 4.37.

Recall that $Mob_1(U^n) = \{\phi|U^n : \phi \in Mob(U^n)\}$, and $Mob_1(U^n)$ is a subgroup of $\mathcal{I}_\eta(U^n)$ by Lemma 4.33. We want to prove $Mob_1(U^n) = \mathcal{I}_\eta(U^n)$.

Recall that, by Theorems 1.12.9 and 1.16, $O^+(M^{n+1})$ is generated by reflections of the form Z_u where u is a space-like unit vector. Also $f \mapsto f|H^n : O^+(M^{n+1}) \to I(H^n)$ is an isomorphism by Theorem 2.3. Furthermore, $\mathcal{X}_\sigma : I(H^n) \to \mathcal{I}_\eta(U^n)$ is an isomorphism by Lemma 4.35. Hence, $\mathcal{I}_\eta(U^n)$ is generated by elements of the form $\mathcal{X}_\sigma(Z_u|H^n)$ where u is a space-like unit vector. Thus, it suffices to prove $\mathcal{X}_\sigma(Z_u|H^n) \in Mob_1(U^n)$ for every spacelike unit vector u.

Let $u \in M^{n+1}$ be a space-like unit vector.

First suppose $u \in E^n \times \{0\}$. Then Lemma 4.38 implies $\mathcal{X}_\sigma(Z_u|H^n) = \hat{Z}_{u,0}|U^n$. (Here we are identifying $E^n \times \{0\}$ with E^n.) Since $Z_{u,0}$ is distance preserving and $Z_{u,0}(0) = 0$, then $\hat{Z}_{u,0}(U^n) = U^n$. Thus $\hat{Z}_{u,0} \in Mob(U^n)$. Hence $\hat{Z}_{u,0}|U^n \in Mob_1(U^n)$. Thus, $\mathcal{X}_\sigma(Z_u|H^n) \in Mob_1(U^n)$.

Now assume $u \notin E^n \times \{0\}$. Then $P_u \cap H^n \in \Gamma^n$ by Theorem 2.15. Since $e_{n+1} \in P_u \Rightarrow e_{n+1} \circ u = 0 \Rightarrow u \in E^n \times \{0\}$, then $e_{n+1} \notin P_u$. Let $x \in P_u \cap H^n$. Then $x \neq e_{n+1}$ and, therefore, $\sigma(x) \neq \sigma(e_{n+1}) = 0$. Lemma 4.40 provides a $\phi \in Mob(U^n)$ such that $\phi(\sigma(x)) = 0$. Lemma 4.33 implies $\phi|U^n \in \mathcal{I}_n(U^n)$. Hence, $\mathcal{X}_\sigma^{-1}(\phi|U^n) = \sigma^{-1} \circ \phi \circ \sigma \in \mathcal{I}(H^n)$. Let $\psi = \mathcal{X}_\sigma^{-1}(\phi|U^n)$. Then $\psi(x) = \sigma^{-1} \circ \phi \circ \sigma(x) = \sigma^{-1}(0) = e_{n+1}$. Since $P_u \cap H^n \in \Gamma^n$ and $\psi \in \mathcal{I}(H^n)$, then $\psi(P_u \cap H^n) \in \Gamma^n$. Also $e_{n+1} = \psi(x) \in \psi(P_u \cap H^n)$.

Theorem 2.15 implies $\varphi(P_u \cap \mathbb{H}^n) = W \cap \mathbb{H}^n$ where W is an n-dimensional vector subspace of $\mathbb{M}^{n+1}$. As we have done before, we extend e_{n+1} to an orthonormal basis $v, v_2, \ldots, v_n, e_{n+1}$ for $\mathbb{M}^{n+1}$ such that $v_2, \ldots, v_n, e_{n+1}$ is an orthonormal basis for W. Then $W = P_v$. Since $v \circ e_{n+1} = 0$, then $v \in \mathbb{E}^n \times \{0\}$. Recall that $Z_v | \mathbb{H}^n \in \mathcal{I}(\mathbb{H}^n)$ by Theorems 1.12 and 2.3. Hence, Lemma 4.38 implies $\chi_o(Z_v|\mathbb{H}^n) = \hat{Z}_{v,0} | \mathbb{U}^n$. Since $\hat{Z}_{v0}(\mathbb{O}) = 0$ and $\hat{Z}_{v0}$ is distance preserving, then $\hat{Z}_{v0}(\mathbb{U}^n) = \mathbb{U}^n$. Thus, $\hat{Z}_{v0} \in \text{Möb}(\mathbb{U}^n)$. Thus, $\hat{Z}_{v0} | \mathbb{U}^n \in \text{Möb}_1(\mathbb{U}^n)$.

Consider $\varphi \circ (Z_u | \mathbb{H}^n) \circ \varphi^{-1} \in \mathcal{I}(\mathbb{H}^n)$. Since the fixed point set of $Z_u | \mathbb{H}^n$ is $P_u \cap \mathbb{H}^n$, then the fixed point set of $\varphi \circ (Z_u | \mathbb{H}^n) \circ \varphi^{-1}$ is $\varphi(P_u \cap \mathbb{H}^n) = P_v \cap \mathbb{H}^n$. [For a function $g: X \to X$, let $\text{Fix}(g)$ denote the fixed point <u>set</u> of g: $\text{Fix}(g) = \{x \in X : g(x) = x\}$. In general, if $f: X \to Y$ is a bijection, then $\text{Fix}(f \circ g \circ f^{-1}) = f(\text{Fix}(g))$.] Then Theorem 2.3½ (pages 4.56–4.57) implies that $\varphi \circ (Z_u|\mathbb{H}^n) \circ \varphi^{-1}$ is either $id_{\mathbb{H}^n}$ or $Z_v|\mathbb{H}^n$. Since $Z_u|\mathbb{H}^n \neq id_{\mathbb{H}^n}$, then $\varphi \circ (Z_u|\mathbb{H}^n) \circ \varphi^{-1} \neq id_{\mathbb{H}^n}$. Thus, $\varphi \circ (Z_u|\mathbb{H}^n) \circ \varphi^{-1} = Z_v|\mathbb{H}^n$. Hence, $Z_u|\mathbb{H}^n = \varphi^{-1} \circ (Z_v|\mathbb{H}^n) \circ \varphi$.

Since $\hat{Z}_{v0}$ and $\phi \in Mob(U^n)$, then $\phi^{-1} \circ \hat{Z}_{v0} \circ \phi \in Mob(U^n)$.

Also

$$\phi^{-1} \circ \hat{Z}_{v0} \circ \phi | U^n = (\phi | U^n)^{-1} \circ (\hat{Z}_{v0} | U^n) \circ (\phi | U^n) =$$

$$(\chi_\sigma(\psi))^{-1} \circ \chi_\sigma(Z_v | \mathbb{H}^n) \circ \chi_\sigma(\psi) =$$

$$\chi_\sigma(\psi^{-1} \circ (Z_v | \mathbb{H}^n) \circ \psi) = \chi_\sigma(Z_u | \mathbb{H}^n).$$

We conclude that $\chi_\sigma(Z_u | \mathbb{H}^n) \in Mob_1(U^n)$, $\square$

<u>Second proof of Theorem 4.37.</u>

<u>This proof is a simplification of the first proof of Theorem 4.37. This simplification is obtained by invoking Theorem 4.34.</u>

<u>As in the preceding proof, it suffices to prove $\chi_\sigma(Z_u | \mathbb{H}^n) \in Mob_1(U^n)$ for every space-like unit vector $u \in M^{n+1}$. So let $u \in M^{n+1}$ be a space-like unit vector.</u>

Let $T = P_u \cap \mathbb{H}^n$. Then $T \in \Gamma^n$ and Theorem 4.34 implies $\sigma(T) = S \cap U^n$

where $S \in \Sigma^n[S^{n-1}]$. Then Corollary 4.21 implies $J_S \in Mob(U^n)$. Thus $J_S|U^n \in Mob_1(U^n)$.

Lemma 4.33 implies $J_S|U^n \in \mathcal{I}_\eta(H^n)$. Since $\chi_\sigma : \mathcal{I}(H^n) \to \mathcal{I}_\eta(U^n)$ is an isomorphism by Lemma 4.35, then

$$\chi_\sigma^{-1}(J_S|U^n) = \sigma^{-1} \circ J_S \circ \sigma \in \mathcal{I}(H^n).$$

Since $Fix(J_S) = S$, then $Fix(J_S|U^n) = S \cap U^n$. Thus, $Fix(\chi_\sigma^{-1}(J_S|U^n)) = \sigma^{-1}(S \cap U^n)$.

~~(See the bracketted remark on page 4.111 in the proof of Theorem 4.37.)~~

Thus, $Fix(\chi_\sigma^{-1}(J_S|U^n)) = T = P_u \cap H^n$ and $\chi_\sigma^{-1}(J_S|U^n) \in \mathcal{I}(H^n)$. It follows by Theorem 2.3½ (page 4.56) that either $\chi_\sigma^{-1}(J_S|U^n) = id_{H^n}$ or $\chi_\sigma^{-1}(J_S|U^n) = Z_u|H^n$.

Since $J_S|U^n \neq id_{U^n}$, then $\chi_\sigma^{-1}(J_S|U^n) \neq id_{H^n}$. Hence, $\chi_\sigma^{-1}(J_S|U^n) = Z_u|H^n$. Therefore,

$\chi_\sigma(Z_u|H^n) = J_S|U^n \in Mob_1(U^n)$. $\square$

Theorem 4.41. A subset of $\mathcal{U}^n$ is a geodesic with respect to the metric $\eta_{\mathcal{U}}$ if and only if it is of the form $S \cap \mathcal{U}^n$ where S is either
a) a line through the origin in $\mathbb{E}^n$, or
b) a circle orthogonal to S^{n-1} in $\mathbb{E}^n$.

Proof Observe that a homeomorphism $\lambda : \mathbb{R} \to (-1,1)$ is defined by

$$\lambda(t) = \frac{e^t - 1}{e^t + 1} = \frac{1 - e^{-t}}{1 + e^{-t}}.$$

(λ is strictly increasing because $\lambda'(t) = \dfrac{2e^t}{(e^t+1)^2} > 0$, $\lim_{t \to \infty} \lambda(t) = 1$ and $\lim_{t \to -\infty} \lambda(t) = -1$.)

A straightforward calculation shows that if v is a unit vector in $\mathbb{E}^n \times \{0\}$, then

$$\sigma \circ \Gamma_{e_{n+1}, v}(t) = \lambda(t) v.$$

If S is a straight line through the origin in $\mathbb{E}^n$, then $S = \mathbb{R}v$ for some unit vector $v \in \mathbb{E}^n \times \{0\}$. Thus $\sigma \circ \Gamma_{e_{n+1}, v}(\mathbb{R}) = \lambda(\mathbb{R}) v = (-1,1)v = S \cap \mathcal{U}^n$. Also $\sigma \circ \Gamma_{e_{n+1}, v}$ is a geodesic in $\mathcal{U}^n$. We conclude that $S \cap \mathcal{U}^n$ is a geodesic in $\mathcal{U}^n$.

Conversely, assume G is a geodesic in U^n such that $0 \in G$. Then $\sigma^{-1}(G)$ is a geodesic in H^n such that $e_{n+1} = \sigma^{-1}(0) \in \sigma^{-1}(G)$. Hence, there is a unit vector v in $E^n \times \{0\}$ such that $\sigma^{-1}(G) = \Gamma_{e_{n+1}, v}(\mathbb{R})$. Hence, $G = \sigma \circ \Gamma_{e_{n+1}, v}(\mathbb{R}) = \lambda(\mathbb{R})v = (-1,1)v$.

Let $S = \mathbb{R}v$. Then S is a straight line through the origin in E^n and $G = S \cap U^n$.

Suppose S is a circle orthogonal to S^{n-1} in E^n. (Then $0 \notin S$ because if S has center c and radius r, then $\|c\| = \sqrt{r^2+1} > r$.) Choose $x \in S \cap U^n$. Then Theorem 4.40 provides a $\phi \in \mathrm{Mob}(U^n)$ such that $\phi(x) = 0$. Since $S \in \Sigma^n_{n-1}$, then Theorem 4.27 implies $\phi(S) \in \Sigma^n_{n-1}$. Thus, $\phi(S)$ is either a straight line or a circle in E^n. Since S is orthogonal to S^{n-1}, $\phi(S^{n-1}) = S^{n-1}$ and ϕ is conformal by Theorem 4.3, then $\phi(S)$ is orthogonal to S^{n-1} by Theorem 4.30. Since $0 = \phi(x) \in \phi(S)$ and no circle in E^n that is orthogonal to S^{n-1} contains 0, then $\phi(S)$ is a straight line through 0 in E^n. Hence, the first part of this proof implies $\phi(S) \cap U^n$ is a geodesic. Since $\phi | U^n \in I_n(U^n)$ by Theorem 4.37, then $\phi^{-1} | U^n \in I_n(U^n)$. Therefore $\phi^{-1}(\phi(S) \cap U^n) = S \cap U^n$ is a geodesic in U^n.

Finally assume G is a geodesic in U^n such that $0 \notin G$. Choose $x \in G$. Theorem 4.40 provides a $\phi \in \mathrm{Mob}(U^n)$ such that $\phi(x) = 0$. Theorem 4.37 implies $\phi|U^n \in I_n(U^n)$. Hence, $\phi(G)$ is a geodesic in U^n. Since $0 = \phi(x) \in \phi(G)$, then the first part of this proof implies $\phi(G) = S \cap U^n$ where S is a straight line through 0 in E^n. Hence, $S \in \Sigma^n_{n-1}$ and S is orthogonal to S^{n-1}. Since $\phi^{-1} \in \mathrm{Mob}(U^n)$, then $\phi^{-1}(S) \in \Sigma^n_{n-1}$ by Theorem 4.27. Hence, $\phi^{-1}(S)$ is a straight line or a circle in E^n. Since S is orthogonal to S^{n-1}, $\phi^{-1}(S^{n-1}) = S^{n-1}$ and ϕ^{-1} is conformal by Theorem 4.3, then $\phi^{-1}(S)$ is orthogonal to S^{n-1} by Theorem 4.30. Note:

$$\phi^{-1}(S) \cap U^n = \phi^{-1}(S \cap U^n) = G.$$

Since $0 \notin G$, then $0 \notin \phi^{-1}(S)$. Since a straight line in E^n that is orthogonal to S^{n-1} must pass through 0, then $\phi^{-1}(S)$ is not a straight line. Thus, $\phi^{-1}(S)$ is a circle orthogonal to S^{n-1} in E^n and $\phi^{-1}(S) \cap U^n = G$. $\square$